

# TECHNICAL RESEARCH REPORT

Approximate Nonlinear Filtering and Its Applications for GPS

by Babak Azimi-Sadjadi, P.S. Krishnaprasad

TR 2001-26



*ISR develops, applies and teaches advanced methodologies of design and analysis to solve complex, hierarchical, heterogeneous and dynamic problems of engineering technology and systems for industry and government.*

*ISR is a permanent institute of the University of Maryland, within the Glenn L. Martin Institute of Technology/A. James Clark School of Engineering. It is a National Science Foundation Engineering Research Center.*

**Web site** <http://www.isr.umd.edu>

<b>Report Documentation Page</b>			Form Approved OMB No. 0704-0188	
<p>Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.</p>				
1. REPORT DATE <b>SEP 2000</b>	2. REPORT TYPE	3. DATES COVERED -		
4. TITLE AND SUBTITLE <b>Approximate nonlinear Filtering and Its Applications for GPS</b>			5a. CONTRACT NUMBER	
			5b. GRANT NUMBER	
			5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)			5d. PROJECT NUMBER	
			5e. TASK NUMBER	
			5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>Army Research Office, PO Box 12211, Research Triangle Park, NC, 27709-2211</b>			8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S)	
			11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>				
13. SUPPLEMENTARY NOTES				
14. ABSTRACT <b>see report</b>				
15. SUBJECT TERMS				
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES <b>28</b>
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>		19a. NAME OF RESPONSIBLE PERSON

# Approximate Nonlinear Filtering and Its Applications for GPS

B. Azimi-Sadjadi and P.S. Krishnaprasad

Institute for Systems Research  
University of Maryland at College Park  
College Park, MD 20742

September, 2000

## Abstract

In this report we address the problem of nonlinear filtering in the presence of integer uncertainty. In the simulation results we show that *particle filtering* is capable of resolving integer ambiguity in the given nonlinear setup. Motivated by these results we introduce particle filtering for an exponential family of densities. We prove that under certain conditions the approximated conditional density converges to the true conditional density. For the case where the conditional density does not lie in an exponential family but stays close to it, we show that under certain assumptions the error of the estimate given by this approximate nonlinear filtering, *projection particle filtering*, is bounded. In the simulation results we show the application of particle filtering to Global Position System (GPS).

## 1 Introduction

GPS provides world wide positioning with acceptable accuracy, if four or more satellites are in view. Although the satellite constellation guarantees availability of four or more satellites (sometimes even nine) world wide, natural or man-made obstacles can easily block the satellites' signal. To overcome this vulnerability, one might think of integrating dead reckoning or Inertial Navigation System (INS) with the GPS [1][2][3]. In this case, the INS or the dead reckoning provide positioning that is adjusted by the GPS.

Using differential GPS allows the user to have a more accurate measurement. In fact, a good portion of the positioning error can be removed from the estimation. This and new technology allow the use of the carrier phase as part of the positioning information. This can increase the accuracy of the estimation to centimeter, or in the static case, to millimeter levels. This can happen only if we are able to estimate the number of full cycles of the carrier phase, which cannot be measured. This problem is called integer ambiguity resolution [4][5] [6][7].

Although carrier phase differential GPS allows for very accurate positioning, it is very sensitive to obstacles that can block satellite signals and cycle slips. A good estimation algorithm should be able to quickly estimate the integer ambiguity on the fly. Most of the algorithms use integer least

---

This research was supported in part by the Army Research Office under the ODDR&E MURI97 Program Grant No. DAAG55-97-1-0114 to the Center for Dynamics and Control of Smart Structures (through Harvard University), and by the National Science Foundation under the Learning and Intelligent Systems Initiative Grant CMS9720334

square methods for this [6][7][4]. In [4] a Kalman filter type of setup is used to estimate the integer ambiguity.

In most of the applications, integrated INS/GPS, dead reckoning/GPS, or vehicle dynamics/GPS, linearization of the dynamics and the GPS observation is the main tool for estimation [6][7][4]. It can be shown [3] that when the number of satellites is below a certain level or the geometry of the current constellation is near singular, the linearization causes the system to be unobservable. In this case, it is important to use a nonlinear setup for the estimation problem. In [3] this case is studied by using an approximation for nonlinear filtering [8][9].

Except for very special cases in nonlinear settings, estimating the state given the observations results in an infinite dimensional filter [10]. Therefore, approximation methods of finite dimension are very appealing. The most widely used approximation filtering method is the Extended Kalman Filter (EKF), which is a heuristic method based on the linearization of the state dynamics and observation near the nominal path [10]. EKF is computationally simple but, the convergence of the conditional distribution to the actual distribution is not guaranteed.

*Projection filtering* is another approximation method [11][12][13][14]. In projection filtering it is assumed that the conditional density of the state of the system can be approximated by a member of parametric family of densities. In this case, estimating the conditional density is equivalent to estimating the parameter of the family. In [11][12][13] the exponential family of densities is chosen as the parametric family. In [14] the approach is different; there a Galerkin approximation is used for solving the Fokker-Planck equation [10].

An entirely different approach to approximate the conditional density was proposed in [8][9]. This method is based on the Monte Carlo method and is called *particle filtering*. In this method, the particles at time  $t_i$  are i.i.d. random vectors that are distributed according to the empirical conditional distribution of the state, given the observations up to time  $t_i$ . These particle/state vectors are used in the state equation to find the values of particles at time  $t_{i+1}$ . Then at time  $t_{i+1}$ , the empirical distribution is evaluated according to the values of the particles. The new observation at time  $t_{i+1}$  is taken into account through Bayes' Rule to calculate the conditional empirical distribution, and this process can be repeated. In [8] it is proved that by a large enough number of particles, one can get an approximate conditional distribution that is arbitrarily close to the true conditional distribution.

In the cases where we have some prior information about the distribution, we should expect to achieve higher performance if we take this information into account. By higher performance, we mean a reduction in the computational cost and an increase in the convergence rate. Here we assume that the conditional distribution has a density in an exponential family of densities, or at least stays close to it in a sense that we will define. Using this assumption, we replace the empirical distribution in [8] with the Maximum Likelihood Estimate (MLE) of the parameters of an exponential density. We call this new method *projection particle filtering*. In Theorem 5.6 we show that if the conditional density of the state given the observations lies in an exponential family of densities then the estimated conditional density converges to the true conditional density in a sense that will be defined. In Theorem 6.7 for the case where the true conditional density stays close to an exponential family of densities we show that the error of the estimate given by projection particle filtering is bounded.

To use nonlinear filtering methods for carrier phase differential GPS, one should be able to include the integer ambiguity resolution in these methods. In this report we present some simulation results which show that particle filtering, with minor modifications, is capable of resolving integer uncertainties present in a problem similar to carrier phase differential GPS. One problem of particle filtering is the need for large number of particles. This problem is even more important for the

cases where integer uncertainty is present. The writers are optimistic that particle filtering for exponential families of densities is more suitable for nonlinear filtering with integer ambiguity.

In this report, Section 2 states the nonlinear filtering problem. In Section 3 we review the results in [11][12][13] on projection filtering. In Section 4 we explain particle filtering and we state the results in [8][9]. In Sections 5 and 6 we introduce a new particle filtering and we state the main results of this report. In Section 7 we apply the particle filtering to a nonlinear system with integer uncertainty and we present the simulation results. In Section 8 we discuss the future research on this subject.

## 2 Nonlinear Filtering, Problem Setup

Filtering problems consist of “estimating” the process  $\{\mathbf{x}_t\}$  (or something about it) given the related process,  $\{\mathbf{y}_t\}$ , which can be observed [15]. The observation is available in an interval, i.e.,  $\{\mathbf{y}_s, 0 \leq s < t\}$  and the function of the state is estimated at time  $t$ . To proceed, we need to give some structure to the concerned processes.

We assume that all stochastic processes are defined on a fixed probability space  $(\Omega, \mathcal{F}, P)$ , and a finite time interval,  $[0, T]$ , on which there is defined an increasing family of  $\sigma$ -fields,  $\{\mathcal{F}_t, 0 \leq t \leq T\}$ . It is assumed that each process,  $\{\mathbf{x}_t\}$ , is adapted to  $\mathcal{F}_t$ , i.e.,  $\{\mathbf{x}_t\}$  is  $\mathcal{F}_t$ -measurable for all  $t$ . We assume that  $\{\mathbf{x}_t\}$  is a vector diffusion process of the form

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{f}_s(\mathbf{x}_s)ds + \int_0^t G_s(\mathbf{x}_s)d\mathbf{w}_s, \quad (1)$$

where  $\mathbf{x}_t \in \mathcal{R}^n$ , and  $\mathbf{w}_t \in \mathcal{R}^q$  is a vector from an independent Brownian motion process; the second integral is in the Ito sense [16], and the function  $\mathbf{f}_t(\cdot)$  and the matrix  $G_t(\cdot)$  have the proper dimensions. The observation,  $\mathbf{y}_t$ , is a discrete time process given as follows:

$$\mathbf{y}_{n\tau} = \mathbf{h}_n(\mathbf{x}_{n\tau}) + \mathbf{v}_n, \quad (2)$$

where  $\mathbf{y}_{n\tau} \in \mathcal{R}^d$ , and  $\mathbf{v}_n \in \mathcal{R}^d$  is a discrete time white Gaussian noise process with zero mean and known covariance matrix. The state dynamics and observation equations can be rewritten formally as follows:

$$\begin{aligned} d\mathbf{x}_t &= \mathbf{f}_t(\mathbf{x}_t)dt + G_t(\mathbf{x}_t)d\mathbf{w}_t, \quad \text{given the distribution of } \mathbf{x}_0 \\ \mathbf{y}_{n\tau} &= \mathbf{h}_n(\mathbf{x}_{n\tau}) + \mathbf{v}_n \end{aligned} \quad (3)$$

The noise processes  $\{\mathbf{w}_t, t \geq 0\}$ , and  $\{\mathbf{v}_n, n = 0, 1, \dots\}$ , and the initial condition  $\mathbf{x}_0$  are assumed to be independent. We use  $Q_t$  and  $R_n$  for the covariance matrices of the processes  $\mathbf{w}_t$  and  $\mathbf{v}_n$ , respectively. We assume that  $R_n$  is invertible for all  $n$ 's. We have the following additional assumptions [17]:

**A 2.1** [local Lipschitz continuity]  $\forall \mathbf{x}, \mathbf{x}' \in B_r$  and  $t \in [0, T]$ , where  $B_r$  is a ball of radius  $r$ , we have

$$\begin{aligned} \|\mathbf{f}_t(\mathbf{x}) - \mathbf{f}_t(\mathbf{x}')\| &\leq k_r \|\mathbf{x} - \mathbf{x}'\|, \quad \text{and} \\ \|G_t(\mathbf{x})Q_tG_t^T(\mathbf{x}) - G_t(\mathbf{x}')Q_tG_t^T(\mathbf{x}')\| &\leq k_r \|\mathbf{x} - \mathbf{x}'\|. \end{aligned} \quad (4)$$

**A 2.2** [Non-Explosion] There exists  $k > 0$  such that

$$\begin{aligned} \mathbf{x}^T \mathbf{f}_t(\mathbf{x}) &\leq k(1 + \|\mathbf{x}\|^2), \quad \text{and} \\ \text{trace}(G_t(\mathbf{x})Q_tG_t^T(\mathbf{x})) &\leq k(1 + \|\mathbf{x}\|^2). \end{aligned} \quad (5)$$

$\forall t \in [0, T]$  and  $\forall \mathbf{x} \in \mathcal{R}^n$ .

Under Assumptions (A2.1) and (A2.2), there exists a unique solution  $\{\mathbf{x}_t, t \in [0, T]\}$  to the state equation, and  $\mathbf{x}_t$  has finite moment of any order [17].

In addition to these, we assume that the probability distribution of the state  $\mathbf{x}_t$ , given the observation up to time  $t$ ,  $\pi_t(d\mathbf{x}) = P(\mathbf{x}_t \in d\mathbf{x} | \mathbf{y}^t)$ , where  $\mathbf{y}^t = \{\mathbf{y}_n, i = 1, \dots, n, n\tau < t\}$ , has a density  $p_t$  with respect to the Lebesgue measure on  $\mathcal{R}^n$ . Then  $\{p_t, t > 0\}$  satisfies the following partial differential equation and updating equations [12]:

$$\begin{aligned}\frac{\partial}{\partial t} p_t &= \mathcal{L}_t^* p_t & n\tau \leq t < (n+1)\tau, \text{ and} \\ p_{n\tau} &= c_n \Psi_n p_{n\tau^-}\end{aligned}\tag{6}$$

where

$$\begin{aligned}\mathcal{L}_t^*(\Phi) &= -\sum_{i=1}^n \frac{\partial}{\partial \mathbf{x}_i} [f_t^i \Phi] + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} [a_t^{ij} \Phi], \\ [a_t^{ij}] &= G_t Q_t G_t^T, \\ \Psi_n(\mathbf{x}) &\triangleq \exp\left(-\frac{1}{2}(\mathbf{y}_{n\tau} - \mathbf{h}_n(\mathbf{x}))^T R_n^{-1}(\mathbf{y}_{n\tau} - \mathbf{h}_n(\mathbf{x}))\right),\end{aligned}$$

and  $c_n$  is a normalizing factor.

Except for the linear Gaussian case, and some very special nonlinear cases, solving System (6) constitutes an infinite dimensional filter [10]. Therefore, for practical problems it is necessary to approximate the conditional density in (6). In the next section, we discuss one of these approximation methods.

### 3 Projection Filtering on Exponential Families of Densities

This section is mainly a review of the results we use from [12]. We start this section with the definition of the exponential family of densities.

**Definition 3.1** Let  $\{c_1, \dots, c_p\}$  be affinely independent<sup>1</sup> scalar functions defined on  $\mathcal{R}^n$ , and assume that the convex set

$$\Theta_0 = \left\{ \theta \in \mathcal{R}^p : \Upsilon(\theta) = \log \int \exp(\theta^T \mathbf{c}(\mathbf{x})) d\mathbf{x} < \infty \right\},$$

has nonempty interior. Then,

$$\begin{aligned}\mathcal{S} &= \{p(\cdot, \theta), \theta \in \Theta\} \\ p(\mathbf{x}, \theta) &:= \exp\left[\theta^T \mathbf{c}(\mathbf{x}) - \Upsilon(\theta)\right],\end{aligned}$$

where  $\Theta \subseteq \Theta_0$  is open, is called an exponential family of probability densities.

We denote by  $\mathcal{S}^{\frac{1}{2}}$  the space of square roots of the densities in  $\mathcal{S}$ , i.e.,  $\mathcal{S}^{\frac{1}{2}} = \{\sqrt{p(\cdot, \theta)}; \theta \in \Theta\}$ . If  $p(\cdot, \theta) \in \mathcal{S}$ , then  $\sqrt{p(\cdot, \theta)} \in L_2$ . The functions  $\frac{1}{2\sqrt{p(\cdot, \theta)}} \frac{\partial p(\cdot, \theta)}{\partial \theta_i}$ ,  $i = 1, \dots, p$  form a basis for the tangent vector space at  $\sqrt{p(\cdot, \theta)}$  to the space  $\mathcal{S}^{\frac{1}{2}}$ , i.e., the tangent space at  $\sqrt{p(\cdot, \theta)}$  is given by [19]:

$$L_{\sqrt{p(\cdot, \theta)}} \mathcal{S}^{\frac{1}{2}} = \text{span} \left\{ \frac{1}{2\sqrt{p(\cdot, \theta)}} \frac{\partial p(\cdot, \theta)}{\partial \theta_1}, \dots, \frac{1}{2\sqrt{p(\cdot, \theta)}} \frac{\partial p(\cdot, \theta)}{\partial \theta_p} \right\}. \tag{7}$$

---

<sup>1</sup> $\{c_1, \dots, c_p\}$  are affinely independent if for distinct points  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{p+1}$ ,  $\sum_{i=1}^{p+1} \lambda_i \mathbf{c}(\mathbf{x}_i) = 0$  and  $\sum_{i=1}^{p+1} \lambda_i = 0$  implies  $\lambda_1 = \lambda_2 = \dots = \lambda_{p+1} = 0$  [18].

The inner product of any two basis elements is defined as follows

$$\begin{aligned} \left\langle \frac{1}{2\sqrt{p(\cdot, \theta)}} \frac{\partial p(\cdot, \theta)}{\partial \theta_i}, \frac{1}{2\sqrt{p(\cdot, \theta)}} \frac{\partial p(\cdot, \theta)}{\partial \theta_j} \right\rangle &= \frac{1}{4} \int \frac{1}{p(\mathbf{x}, \theta)} \frac{\partial p(\mathbf{x}, \theta)}{\partial \theta_i} \frac{\partial p(\mathbf{x}, \theta)}{\partial \theta_j} d\mathbf{x} \\ &= \frac{1}{4} g_{ij}(\theta) \end{aligned} \quad (8)$$

It can be easily seen that  $g(\theta) = (g_{ij}(\theta)) = (E[c_i c_j] - E[c_i]E[c_j])$  is the Fisher information matrix of  $p(\cdot, \theta)$ .

Any member of  $L_2$  can be projected to the tangent space  $L_{\sqrt{p(\cdot, \theta)}} \mathcal{S}^{\frac{1}{2}}$  according to the following projection formula

$$\begin{aligned} \Pi_\theta : L_2 \supseteq V &\rightarrow L_{\sqrt{p(\cdot, \theta)}} \mathcal{S}^{\frac{1}{2}} \\ v &\rightarrow \sum_{i=1}^p \sum_{j=1}^p 4g^{ij}(\theta) \left\langle v, \frac{1}{2\sqrt{p(\cdot, \theta)}} \frac{\partial p(\cdot, \theta)}{\partial \theta_j} \right\rangle \frac{1}{2\sqrt{p(\cdot, \theta)}} \frac{\partial p(\cdot, \theta)}{\partial \theta_i}. \end{aligned} \quad (9)$$

Projection filtering seeks a solution  $p_t$  for (6) that lies in  $\mathcal{S}$ . Of course, this solution is only an exponential density, but we hope, by choosing the proper family, to keep the approximation error small (in the  $L_2$  sense).

If we consider the square root of the density in (6), we get

$$\frac{\partial \sqrt{p_t}}{\partial t} = \frac{1}{2\sqrt{p_t}} \frac{\partial p_t}{\partial t} = \frac{1}{2\sqrt{p_t}} \mathcal{L}_t^* p_t. \quad (10)$$

Define  $\alpha_{t, \theta} = \frac{\mathcal{L}_t^* p_t(\cdot, \theta)}{p_t(\cdot, \theta)}$ . We assume that for all  $\theta \in \Theta$  and all  $t \geq 0$ ,  $E_{p(\cdot, \theta)}\{|\alpha_{t, \theta}|^2\} < \infty$ , which implies that  $\frac{\mathcal{L}_t^* p_t(\cdot, \theta)}{\sqrt{p_t(\cdot, \theta)}}$  is a vector in  $L_2$  for all  $\theta \in \Theta$  and all  $t \geq 0$  [12].

Now assume that in equation (10), for  $\{\sqrt{p_t}, t \geq t_0\}$ , starting at time  $n\tau$  from the initial condition,  $\sqrt{p_{n\tau}} = \sqrt{p(\cdot, \theta_{n\tau})} \in \mathcal{S}^{\frac{1}{2}}$  for some  $\theta_{n\tau} \in \Theta$ . Under these assumptions, the right hand side of (10) is in  $L_2$ , which can be projected into the finite dimensional tangent vector space  $L_{\sqrt{p(\cdot, \theta_{n\tau})}} \mathcal{S}^{\frac{1}{2}}$ . The propagation part of the projection filter for the exponential family,  $\mathcal{S}$ , in the interval  $[n\tau, (n+1)\tau]$ , is defined as the solution to the following differential equation in the same interval:

$$\frac{\partial \sqrt{p_t(\cdot, \theta_t)}}{\partial t} = \Pi_{\theta_t} \frac{\mathcal{L}_t^* p_t(\cdot, \theta_t)}{2\sqrt{p_t(\cdot, \theta_t)}}. \quad (11)$$

We also assume that  $\mathbf{h}_n(\mathbf{x})$  in equation (2) is time invariant, i.e.,  $\mathbf{h}_n(\mathbf{x}) = \mathbf{h}(\mathbf{x})$ , and the components of  $\mathbf{h}(\mathbf{x})$ ,  $h^i(\mathbf{x})$ , and  $\|\mathbf{h}(\mathbf{x})\|_{R^{-1}}^2$  are linear combinations of  $c_i(\mathbf{x})$ ,  $i = 1, \dots, p$ :

$$\frac{1}{2} \|\mathbf{h}(\mathbf{x})\|_{R^{-1}}^2 = \sum_{i=1}^p \lambda_i^0 c_i(\mathbf{x}) \quad \text{and} \quad h^k(\mathbf{x}) = \sum_{i=1}^p \lambda_i^k c_i(\mathbf{x}), \quad k = 1, \dots, d \quad (12)$$

where  $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T A \mathbf{x}}$ . Then, if  $\mathbf{v}_n$  is stationary with the covariance matrix  $R_n = R$ , the likelihood function  $\Psi_n(n)$  can be written as follows:

$$\begin{aligned} \Psi_n(\mathbf{x}) &= \exp(-\frac{1}{2}(\mathbf{y}_{n\tau}^T R^{-1} \mathbf{y}_{n\tau})) \exp(-\frac{1}{2}(\mathbf{h}^T(\mathbf{x}) R^{-1} \mathbf{h}(\mathbf{x})) + (\mathbf{y}_{n\tau}^T R^{-1} \mathbf{h}(\mathbf{x}))) \\ &= A_n \exp\left(-\sum_{i=1}^d \lambda_i^0 c_i(\mathbf{x}) + \sum_{k=1}^p \left(\sum_{i=1}^p \lambda_i^k z_{n\tau}^k\right) c_i(\mathbf{x})\right), \end{aligned} \quad (13)$$

where  $\mathbf{z}_{n\tau} = \mathbf{y}_{n\tau}^T R^{-1}$ , and  $A_n$  is a constant depending on  $\mathbf{y}_{n\tau}$ . Therefore, the coefficient  $\Psi_n(\mathbf{x})$  is a member of exponential family of densities. This family is closed under multiplication. Using all of these facts, we can present the following theorem [12]:

**Theorem 3.2 [Brigo 1996]** For system (3), where  $\mathbf{w}_t$  is a Brownian motion process with covariance  $Q_t$  and  $\mathbf{v}_i$  is a white Gaussian noise with covariance  $R$ , we assume (A2.1) and (A2.2) to be true. We also assume that  $\frac{1}{2}\|\mathbf{h}(\mathbf{x})\|_{R^{-1}}^2 = \sum_{i=1}^p \lambda_i^0 c_i(\mathbf{x})$ ,  $h^k(\mathbf{x}) = \sum_{i=1}^p \lambda_i^k c_i(\mathbf{x})$ , for  $k = 1, \dots, d$ , and  $E_{p(\cdot, \theta)} \|\frac{\mathcal{L}_t^* p(\cdot, \theta)}{p(\cdot, \theta)}\|^2 < \infty$ ,  $\forall \theta \in \Theta$ ,  $\forall t \geq 0$ . Then for all  $\theta \in \Theta$ , and all  $t \geq 0$ ,  $\Pi_\theta \frac{\mathcal{L}_t^* p(\cdot, \theta)}{\sqrt{p(\cdot, \theta)}}$  is a vector on the exponential manifold  $\mathcal{S}^{\frac{1}{2}}$ . The projection filter density,  $p_t^\Pi = p_t(\cdot, \theta_t)$  is described by

$$\begin{aligned} \frac{\partial \sqrt{p_t(\cdot, \theta_t)}}{\partial t} &= \Pi_{\theta_t} \frac{\mathcal{L}_t^* p_t(\cdot, \theta_t)}{2\sqrt{p(\cdot, \theta_t)}}, & n\tau \leq t < (n+1)\tau \\ p_{n\tau}(\cdot, \theta_{n\tau}) &= c_n \Psi_n(\mathbf{y}_{n\tau}) p_{n\tau^-}(\cdot, \theta_{n\tau^-}), \end{aligned}$$

and the projection filter parameter satisfies the following combined differential and stochastic difference equations:

$$\begin{aligned} g(\theta_t) d\theta_t &= E_{\theta_t} \{\mathcal{L}_t \mathbf{c}\} dt, & n\tau \leq t < (n+1)\tau, \\ \theta_{n\tau} &= \theta_{n\tau^-} - \lambda_0^0 + \sum_{k=1}^d \lambda_0^k z_n^k, \end{aligned}$$

where

$$\mathcal{L}_t = \sum_{i=1}^n f_t^i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_t^{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

and  $\lambda_0^i = [\lambda_1^i, \dots, \lambda_p^i]^T$ ,  $i = 0, \dots, d$ , and  $z_n^k$  is the  $k$ th component of  $\mathbf{z}_{n\tau}^T = R^{-1} \mathbf{y}_{n\tau}$ .

Henceforth, we shall use  $E_\theta$  and  $E_{p(\cdot, \theta)}$ ,  $\theta_{n\tau}$  and  $\theta_n$ , and  $p_{n\tau}$  and  $p_n$ , interchangeably, respectively.

**Remark:** The differential equation for  $\theta_t$  is an ordinary differential equation with the vector field  $g(\theta_t)^{-1} E_{\theta_t} \{\mathcal{L}_t \mathbf{c}\}$ . This vector field should be computed analytically. If the analytical computation of this vector field is not possible an off-line numerical computation should be carried.

As can be seen from the statement of the theorem, the calculation of the conditional probability density is reduced to the calculation of the parameter of an exponential family. But, solving the differential equation in the theorem is not an easy task. At each moment  $g(\theta_t)$  and  $E_{\theta_t} \{\mathcal{L}_t \mathbf{c}\}$  need to be calculated. This imposes a heavy computational load. In this report, we introduce a Monte Carlo method to calculate the parameter of the exponential family with a more affordable computational load.

Although projection filtering gives a better solution than EKF, there is no known error bound with which we can compare the distance between the real density and the density given by the projection filter. In the next section we review particle filtering as an alternative to optimal nonlinear filtering.

**Remark :** The assumption on  $\mathbf{h}_n(\cdot)$  and  $R_n$  in this are made only to ensure that  $\Psi_n(\cdot)$  is in the family of exponential densities. These assumptions can be relaxed if  $\Psi_n(\cdot)$  is guaranteed to stay in the family.

## 4 Particle Filtering

Consider either the continuous dynamics and discrete observation in (3) or the discrete case,

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{f}_n(\mathbf{x}_n) + G_n(\mathbf{x}_n) \mathbf{w}_n, \text{ given the distribution of } \mathbf{x}_0 \\ \mathbf{y}_n &= \mathbf{h}_n(\mathbf{x}_n) + \mathbf{v}_n. \end{aligned} \tag{14}$$

We assume that in both cases, the initial distribution for  $\mathbf{x}_0$  is given. The propagation of the conditional density, at least conceptually, can be calculated as follows [10]:

- Step 1 . Initialization:

$$p_0(\mathbf{x}_0|\mathbf{y}_0) = p(\mathbf{x}_0).$$

- Step 2 . Diffusion:

$$p_{(n+1)^-}(\mathbf{x}_{n+1}|\mathcal{Y}_n) = \int p(\mathbf{x}_{n+1}|\mathbf{x}_n)p_n(\mathbf{x}_n|\mathcal{Y}_n)d\mathbf{x}_n,$$

where  $\mathcal{Y}_n = \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ .

- Step 3 . Bayes' rule update:

$$p_{(n+1)}(\mathbf{x}_{n+1}|\mathcal{Y}_{n+1}) = \frac{p(\mathbf{y}_{n+1}|\mathbf{x}_{n+1})p_{(n+1)^-}(\mathbf{x}_{n+1}|\mathcal{Y}_n)}{\int p(\mathbf{y}_{n+1}|\mathbf{x}_{n+1})p_{(n+1)^-}(\mathbf{x}_{n+1}|\mathcal{Y}_n)d\mathbf{x}_{n+1}} ,$$

- Step 4 .  $n \leftarrow n + 1$ ; go to Step (2).

The conditional density given by the above steps is exact, but in general it can be viewed as an infinite dimensional filter, thus, not implementable. Particle filtering, in brief, is an approximation method that mimics the above calculations with a finite number of operations using the Monte Carlo method. The procedure for particle filtering is as follows [20, 8]:

#### **Algorithm 4.1 Particle Filtering**

- Step 1 . Initialization

◊ Sample  $\mathbf{x}_0^1, \dots, \mathbf{x}_0^N$ ,  $N$  i.i.d. random vectors with the initial distribution  $P_0(\mathbf{x})$ .

- Step 2 . Diffusion

◊ Find  $\hat{\mathbf{x}}_{n+1}^1, \dots, \hat{\mathbf{x}}_{n+1}^N$  from the given  $\mathbf{x}_n^1, \dots, \mathbf{x}_n^N$ , using the dynamic rules:

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t)dt + G_t(\mathbf{x}_t)d\mathbf{w}_t, \quad n\tau \leq t < (n+1)\tau$$

or

$$\mathbf{x}_{n+1} = \mathbf{f}_n(\mathbf{x}_n) + G_n(\mathbf{x}_n)\mathbf{v}_n.$$

- Step 3 . Find the empirical distribution

$$P_{(n+1)^-}^N(\mathbf{x}) = \frac{1}{N} \sum_{j=1}^N \delta_{\hat{\mathbf{x}}_{n+1}^j}(\mathbf{x})$$

- Step 4 . Use Bayes' Rule

$$P_{(n+1)}^N(\mathbf{x}) = \frac{\frac{1}{N} \sum_{j=1}^N \delta_{\hat{\mathbf{x}}_{n+1}^j}(\mathbf{x}) \cdot \Psi_{n+1}(\mathbf{x})}{\frac{1}{N} \sum_{j=1}^N \delta_{\hat{\mathbf{x}}_{n+1}^j}(\hat{\mathbf{x}}_{n+1}^j) \cdot \Psi_{n+1}(\hat{\mathbf{x}}_{n+1}^j)}$$

- Step 5 . Resample

- ◊ Sample  $\mathbf{x}_{n+1}^1, \dots, \mathbf{x}_{n+1}^N$  according to  $P_{n+1|n+1}^N(\mathbf{x})$
- Step 6 .  $n \leftarrow n + 1$ ; go to Step (2).

where  $\delta_{\mathbf{v}}(\mathbf{w}) = 1$  if  $\mathbf{w} = \mathbf{v}$  and 0 otherwise, and  $\Psi_n(\mathbf{x})$  is the conditional density of the observation  $\mathbf{y}_n$  given the state  $\mathbf{x}$ .

It is customary to call  $\mathbf{x}_n^1, \dots, \mathbf{x}_n^N$  particles. In the next few lines, we try to explain in words the evolution of these particles using the above algorithm.

Let  $\hat{\mathbf{x}}_n^1, \dots, \hat{\mathbf{x}}_n^N$  be the distinct particles at time  $n$  before incorporating the observation at time  $n$ . The probability of each particle is  $\frac{1}{N}$ , that is, is uniformly distributed. After using the observations, the conditional probability of each particle changes. Some will have small, and some large probabilities. Therefore, in the process of resampling, it is very likely that some particles will never be used and instead some other particles (with high probabilities) will be sampled more than once. Therefore, after resampling, some particles have repeated versions, but in the diffusion phase they go through different paths and at the end of the diffusion phase, it is very likely, we would have  $N$  distinct particles. This automatically makes the approximation one of better resolution in the areas where the probability is higher.

In [8] it is proved under some conditions that

$$\lim_{N \rightarrow \infty} E \left( \left| \frac{1}{N} \sum_{i=1}^N f(\hat{\mathbf{x}}_n^i) - E_{P_n}(f(\mathbf{x})) \right| \right) = 0 \quad (15)$$

for every bounded Borel test function,  $f(\cdot)$ .

One problem in using the particle filtering method is the computational cost. In particular, for a high dimensional system, getting reasonable accuracy means using a large  $N$ , which results in a heavy computational cost. In the next section, we propose a method that can reduce the number of particles for a certain class of problems.

## 5 Particle Filtering for Exponential Families of Densities

In the previous section, we saw two approximation methods for nonlinear filtering. In the particle filtering method, we saw that the conditional distribution is approximated by the empirical distribution. Unlike the empirical distribution, in most cases, the actual conditional distribution is smooth. Intuition suggests that if we have prior knowledge of some properties of the distribution, we can improve on the quality of the estimates over just using the empirical distribution. In this section first, we assume that the conditional density lies in an exponential family of densities. We will see that with this assumption, we can show the convergence of the approximated density to the actual one. Later, we relax this assumption and we only require that the conditional density stay close to the exponential family of densities. We prove that the error of the estimate for the latter case is bounded.

For System (3), we assume that the probability density of  $\mathbf{x}_t$ , given the observation, is in a family of exponential densities  $\mathcal{S}$ <sup>2</sup>.

With this assumption, the proposed algorithm is as follows:

**Algorithm 5.1** *Particle Filtering for an Exponential Family of Densities.*

---

<sup>2</sup>This assumption is rather strong. We will drop this assumption later, and we will only assume that there exists a known family of densities that approximates the real density well, i.e., with acceptable accuracy.

- Step 1 . Initialization

◊ Sample  $\mathbf{x}_0^1, \dots, \mathbf{x}_0^N$ ,  $N$  i.i.d. random vectors with the density,  $p_0(\mathbf{x})$ .

- Step 2 . Diffusion

◊ Find  $\hat{\mathbf{x}}_{n+1}^1, \dots, \hat{\mathbf{x}}_{n+1}^N$  from the given  $\mathbf{x}_n^1, \dots, \mathbf{x}_n^N$ , using the dynamic rule:

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t)dt + G_t(\mathbf{x}_t)d\mathbf{w}_t, \quad i\tau \leq t < (i+1)\tau$$

- Step 3 . Find the MLE of  $\hat{\theta}_{(n+1)^-}$  given  $\hat{\mathbf{x}}_{n+1}^1, \dots, \hat{\mathbf{x}}_{n+1}^N$  [21]

$$\hat{\theta}_{(n+1)^-} = \arg \max_{\theta} \prod_{i=1}^N \exp(\theta^T \mathbf{c}(\hat{\mathbf{x}}_{n+1}^i) - \Upsilon(\theta))$$

- Step 4 . Use Bayes' Rule

$$p(\mathbf{x}, \hat{\theta}_{(n+1)}) = \frac{\exp\left(\hat{\theta}_{(n+1)^-}^T \mathbf{c}(\mathbf{x}) - \Upsilon(\hat{\theta}_{(n+1)^-})\right) \Psi_{n+1}(\mathbf{x})}{\int \exp\left(\hat{\theta}_{(n+1)^-}^T \mathbf{c}(\mathbf{x}) - \Upsilon(\hat{\theta}_{(n+1)^-})\right) \Psi_{n+1}(\mathbf{x}) d\mathbf{x}}$$

- Step 5 . Resample

◊ Sample  $\mathbf{x}_{n+1}^1, \dots, \mathbf{x}_{n+1}^N$  according to  $p(\mathbf{x}, \hat{\theta}_{n+1})$ .

- Step 6 .  $n \leftarrow n + 1$ ; go to Step (2).

To generate  $\mathbf{x}_{n+1}^1, \dots, \mathbf{x}_{n+1}^N$ , a Gibbs sampler can be used [22]. This brings an extra computational cost, which should be taken into account when choosing Algorithm 5.1 over Algorithm 4.1.

It is instructive to discuss the structure of the ML estimator. We are going to use this structure for the proof of convergence.

Let  $\hat{\mathbf{x}}_n^1, \dots, \hat{\mathbf{x}}_n^N$  be the value of the particles right before the measurement at time  $n$ . The MLE of  $\theta_n$ ,  $\hat{\theta}_n$ , satisfies the first order necessary condition

$$\sum_{i=1}^N c_j(\hat{\mathbf{x}}_n^i) - N \frac{\int_{\mathbf{x}} c_j(\mathbf{x}) \exp(\hat{\theta}_n^T \mathbf{c}(\mathbf{x})) d\mathbf{x}}{\int_{\mathbf{x}} \exp(\hat{\theta}_n^T \mathbf{c}(\mathbf{x})) d\mathbf{x}} = 0.$$

Therefore, we get

$$\frac{1}{N} \sum_{i=1}^N c_j(\hat{\mathbf{x}}_n^i) = E_{\hat{\theta}_n}(c_j(\mathbf{x})), \quad \text{for } j = 1, \dots, p. \quad (16)$$

Equation (16) says that the sample average of  $c_j(\mathbf{x})$  and its probabilistic average, evaluated at  $\hat{\theta}_n$ , should be equal. The MLE of  $\theta$  is the solution to the system of equations in (16). Let  $F_j(\theta)$  be as follows:

$$F_j(\theta) = \frac{1}{N} \sum_{i=1}^N c_j(\hat{\mathbf{x}}_n^i) - \frac{\int c_j(\mathbf{x}) \exp(\theta^T \mathbf{c}(\mathbf{x})) d\mathbf{x}}{\int \exp(\theta^T \mathbf{c}(\mathbf{x})) d\mathbf{x}}, \quad j = 1, \dots, p.$$

For simplicity we drop the index  $n$  from  $\theta_n$ . It is easy to see that

$$-\frac{\partial F_i}{\partial \theta_j} = E_\theta(c_i(\mathbf{x})c_j(\mathbf{x})) - E_\theta(c_i(\mathbf{x}))E_\theta(c_j(\mathbf{x})).$$

This shows that  $(-\frac{\partial F_i}{\partial \theta_j})_{i,j} = g(\theta)$ , where  $g(\theta)$  is the Fisher information matrix of the exponential density at  $\theta$ . Since  $c_i(\mathbf{x}), i = 1, \dots, p$  are affinely independent  $g(\theta) > 0, \forall \theta \in \Theta$ . Therefore, (16) is the necessary and sufficient condition for optimality.

In the next few pages, we prove the convergence of the MLE of  $\theta_n, \hat{\theta}_n$ , to  $\theta_n$  in the mean square sense.

In each iteration the proposed algorithm starts from the density  $p_{\hat{\theta}_t}(\mathbf{x}_t|\mathbf{y}^t)$ ,  $t = \tau n$ , where  $\hat{\theta}_t$  is the best estimate  $\theta_t$  according to the algorithm. After a full iteration, the algorithm yields  $\hat{\theta}_{t+1}$  which is the best estimate of  $\theta_{t+1}$ . The error in  $\hat{\theta}_{t+1}$  is a combination of the series of possible errors for which we want to find an upper bound. The first source of error is the error in  $\hat{\theta}_t$ , which will propagate even if no other error is considered. The other source comes from the fact that in each iteration new particles are resampled based on the estimated density which is different from the actual density. Finally, the last source of error comes from the discretization of the stochastic dynamics of the system. We want to emphasize that here we assume  $\Psi_n(\mathbf{x}) = \exp(-\frac{1}{2}(\mathbf{y}_{n\tau} - \mathbf{h}_n(\mathbf{x}_{n\tau}))^T R_n^{-1}(\mathbf{y}_{n\tau} - \mathbf{h}_n(\mathbf{x}_{n\tau})))$  lies in the chosen family of densities. Therefore, no other error is added to the estimate because of the Bayes' correction.

We recall the following fact [21]:

**Fact 5.2** *For the family of densities  $\mathcal{S}$  with probability density*

$$p(\mathbf{x}, \theta) = \exp(\theta^T \mathbf{c}(\mathbf{x}) - \Upsilon(\theta)),$$

*the Fisher information matrix  $g(\theta) = (E(c_i(\mathbf{x})c_j(\mathbf{x})) - E(c_i(\mathbf{x}))E(c_j(\mathbf{x})))_{i,j}$  is positive definite. Also the log likelihood function*

$$l(\theta) = \theta^T \mathbf{C}(\mathbf{x}) - \Upsilon(\theta),$$

*is strictly concave. Therefore, for*

$$c_j(\mathbf{x}) = E_\theta[c_j(\mathbf{x})], \quad j = 1, \dots, p,$$

*if a solution exists<sup>3</sup>, it is unique. In addition if  $\mathbf{x}_1, \dots, \mathbf{x}_N$  are  $N$  i.i.d. random variables distributed according to  $p(\mathbf{x}, \theta)$ , then the MLE of  $\theta, \hat{\theta}_N$ , is asymptotically normal, i.e.*

$$\begin{aligned} \hat{\theta}_N &= \arg \max_{\theta} \prod_{i=1}^N p(\mathbf{x}_i, \theta), \\ \sqrt{N}(\hat{\theta}_N - \theta) &\sim \mathcal{N}(0, g^{-1}(\theta)). \end{aligned}$$

Using this fact, it is easy to see that

$$E \left( \left\| \hat{\theta}_N - \theta \right\|^2 \right) = \frac{1}{N} \text{trace}(g^{-1}(\theta)),$$

therefore, when  $N \rightarrow \infty$ ,  $\hat{\theta}_N \rightarrow \theta$  in the m.s. sense. On the other hand,  $\hat{\theta}_N$  is the solution to (16). Using the strong law of large numbers [23], when  $N \rightarrow \infty$  the LHS in (16) goes to

---

<sup>3</sup>In [18] it is shown that if  $N > p$ , the solution exists almost surely.

$E_\theta(c_j(\mathbf{x}))$ ,  $j = 1, \dots, p$ , with probability one. In other words, the solution to (16) when the LHS is the exact  $E_\theta(c_j(\mathbf{x}))$ ,  $j = 1, \dots, p$ , gives the exact solution for  $\theta$ . Using this argument, one can expect that by finding a good estimate of the left hand side of (16), a good estimate of  $\theta$  is accessible. In each iteration of the algorithm presented in this section the estimate of the LHS of (16) is found by using the Monte Carlo method and the approximate solution for the stochastic differential equation (3).

To approximate the solution to the stochastic differential equation (3), we employ the method used in [24]. In the following, we review this method briefly. The stochastic differential equation in (3) can be rewritten as follows:

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t) dt + \sum_{r=1}^q \mathbf{g}_t^r(\mathbf{x}_t) dw_t^r, \quad (17)$$

where  $\mathbf{g}_t^r(\cdot)$  is the  $r^{th}$  column of the matrix  $G_t(\cdot)$ , and  $w_t^r$  is the  $r^{th}$  component of  $\mathbf{w}_t$ . We introduce the operators

$$\begin{aligned} \Lambda_r u &= \left( \mathbf{g}_r, \frac{\partial}{\partial \mathbf{x}} \right) u, \\ Lu &= \left( \frac{\partial}{\partial t} + \left( \mathbf{f}, \frac{\partial}{\partial \mathbf{x}} \right) + \frac{1}{2} \sum_{r=1}^q \sum_{i=1}^n \sum_{j=1}^n g_i^r g_j^r \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \right) u, \end{aligned}$$

where  $\left( \mathbf{a}, \frac{\partial}{\partial \mathbf{x}} \right) = \sum_{i=1}^n a_i \frac{\partial}{\partial \mathbf{x}_i}$ . Then, the approximate solution for the stochastic differential equation can be written as follows:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{x}_k + \sum_{r=1}^q g_{t_k}^r \xi_k^r h^{\frac{1}{2}} + \mathbf{f}_{t_k} h + \sum_{r=1}^q \sum_{i=1}^q (\Lambda_r \mathbf{g}^r)_{t_k} \xi_k^{ir} h + \\ &\quad \frac{1}{2} \sum_{r=1}^q (L \mathbf{g}^r + \Lambda_r \mathbf{f})_{t_k} \xi_k^r h^{\frac{3}{2}} + (L \mathbf{f})_{t_k} \frac{h^2}{2}, \end{aligned} \quad (18)$$

where  $h$  is the step size and the coefficients  $\mathbf{g}_{t_k}^r$ ,  $\mathbf{f}_{t_k}$ ,  $(\Lambda_i \mathbf{g}^r)_{t_k}$ , etc., are computed at the point  $(t_k, \mathbf{x}_k)$ . Also, the sets of random variables  $\xi_k^r$ ,  $\xi_k^{ir}$  are independent for distinct  $k$  and can, for each  $k$ , be modeled as follows:

$$\xi^{ij} = \frac{1}{2} \xi^i \xi^j - \frac{1}{2} \gamma_{ij} \zeta^i \zeta^j, \quad \gamma_{ij} = \begin{cases} -1 & , i < j \\ 1 & , i \geq j \end{cases}.$$

and  $\xi^i$  and  $\zeta^j$  are independent random variables satisfying

$$\begin{aligned} E\xi_i &= E\xi_i^3 = E\xi_i^5 = 0, & E\xi_i^2 &= 1, & E\xi_i^4 &= 3, \\ E\zeta_j &= E\zeta_j^3 = 0, & E\zeta_j^2 &= \zeta_j^4 = 1. \end{aligned}$$

In particular,  $\xi_i$  can be modeled by the law  $P(\xi = 0) = \frac{2}{3}$ ,  $P(\xi = \sqrt{3}) = \frac{1}{6}$ , and  $P(\xi = -\sqrt{3}) = \frac{1}{6}$ , and  $\zeta_j$  can be modeled by  $P(\zeta = -1) = P(\zeta = 1) = \frac{1}{2}$ .

**Definition 5.3** We say that a function  $u(\cdot)$  belongs to the class  $\mathcal{F}$ , written as  $u \in \mathcal{F}$ , if we can find constants,  $k > 0$ , and  $\kappa > 0$ , such that for all  $\mathbf{x} \in \mathcal{R}^n$ , the following inequality holds:

$$\|u(\mathbf{x})\| \leq k(1 + \|\mathbf{x}\|^\kappa).$$

Before we present our results we need to specify the probability space in which the random variables are defined. As we mentioned before, the stochastic process associated to the dynamics and the observation equation are defined on a fixed probability space  $(\Omega, \mathcal{F}, P)$ , the expectation associated to this probability space is denoted by  $E$ . In Algorithm 5.1 the generated particles form a Markov process. Similar to section 2.2 of [8] we denote the probability space associated to this process by  $(\Omega', \mathcal{F}', P'_{[y]})$ . The subindex  $y$  is used to emphasize that the probability measure is conditioned on the observation  $y$ . Another set of random variables,  $\xi^i, \zeta^i$ , are defined for the approximation of the stochastic differential equation (17). We denote the probability space associated to these random variables by  $(\Omega'', \mathcal{F}'', P'')$ . The expectation associated to this process is denoted by  $E''$ . Finally we define  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ , where  $\tilde{\Omega} = \Omega \times \Omega' \times \Omega''$  and  $\tilde{\mathcal{F}} = \mathcal{F} \times \mathcal{F}' \times \mathcal{F}''$ . For every  $\tilde{\omega} \in \tilde{\Omega}$  we define  $\tilde{\omega} = (\omega, \omega', \omega'')$ , then for every  $A \in \mathcal{F}$ ,  $B \in \mathcal{F}'$ , and  $C \in \mathcal{F}''$  we define the probability measure  $\tilde{P}(A \times B \times C) = \int_A \left( \int_C P'_{[Y]}(B) dP''(\omega'') \right) dP(\omega)$ . The expectation with respect the probability measure  $\tilde{P}$  is denoted by  $\tilde{E}$ .

The following theorem summarizes the weak approximation results for (18).

**Theorem 5.4** [ Milstein [24]] Suppose (A2.1) from Section (2), and suppose that the functions  $\mathbf{f}(\cdot)$ ,  $\mathbf{g}^r(\cdot)$ ,  $r = 1, \dots, q$  together with the partial derivatives of sufficiently high order, belong to class  $\mathcal{F}$ . Also, suppose that the functions  $\Lambda_i \mathbf{g}^r$ ,  $L \mathbf{g}^r$ ,  $\Lambda_r \mathbf{f}$ , and  $L \mathbf{f}$  grow at most as a linear function in  $\|\mathbf{x}\|$ . Then, if the function  $u(\cdot)$  and all its derivatives up to order 6 belong to class  $\mathcal{F}$ , the approximation (18) has the order of accuracy 2, in the sense of weak approximation, i.e.,

$$\|\tilde{E}u(\mathbf{x}_{0,\mathbf{x}_0}(t_k)) - \tilde{E}u(\hat{\mathbf{x}}_{0,\mathbf{x}_0}(t_k))\| \leq Kh^2, \quad t_k \in [0, T],$$

where  $K$  is a constant and  $\mathbf{x}_{0,\mathbf{x}_0}(\cdot)$  and  $\hat{\mathbf{x}}_{0,\mathbf{x}_0}(\cdot)$  are the exact and approximate solutions for the stochastic differential equation, respectively.

The Monte Carlo approximation of  $\tilde{E}u(\mathbf{x}_{0,\mathbf{x}_0}(t_k))$  brings another error term. The combination of these errors can be expressed as follows:

$$\begin{aligned} & \left\| \tilde{E}u(\mathbf{x}_{0,\mathbf{x}_0}(t_k)) - \frac{1}{N} \sum_{i=1}^N u(\hat{\mathbf{x}}_{0,\mathbf{x}_0^i}(t_k)) \right\| \leq \\ & \left\| \tilde{E}u(\mathbf{x}_{0,\mathbf{x}_0}(t_k)) - \tilde{E}u(\hat{\mathbf{x}}_{0,\mathbf{x}_0}(t_k)) \right\| + \left\| \tilde{E}u(\hat{\mathbf{x}}_{0,\mathbf{x}_0}(t_k)) - \frac{1}{N} \sum_{i=1}^N u(\hat{\mathbf{x}}_{0,\mathbf{x}_0^i}(t_k)) \right\|. \end{aligned}$$

If the variance of  $u(\hat{\mathbf{x}}_{0,\mathbf{x}_0}(t_k))$  is bounded, we have

$$\tilde{E} \left\| \tilde{E}u(\mathbf{x}_{0,\mathbf{x}_0}(t_k)) - \frac{1}{N} \sum_{i=1}^N u(\hat{\mathbf{x}}_{0,\mathbf{x}_0^i}(t_k)) \right\| \leq Kh^2 + \frac{k'}{N^{1/2}}, \quad (19)$$

where  $K$  and  $k'$  are constants, and  $h$  is the step size for the approximation of the solution of the stochastic differential equation.

The next lemma relates the approximate solution to the stochastic differential equation and the estimate of the parameter  $\theta$ . This lemma is the main building block for our result in this section.

**Lemma 5.5** *For the stochastic differential equation*

$$d\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_t) dt + G_t(\mathbf{x}_t) d\mathbf{w}_t, \quad \mathbf{x}_0, \quad t \in [0, t_f],$$

assume that  $\mathbf{f}_t(\cdot)$ ,  $G_t(\cdot)$  are such that for the Brownian motion,  $\mathbf{w}_t$ , the probability density of the state  $\mathbf{x}_t$  lies in the family  $\mathcal{S}$  for  $\Theta$  bounded, with  $g(\theta) \geq \vartheta I$  for some  $\vartheta > 0$ . We also assume the conditions in Fact 5.2 and in Theorem 5.4 with  $\mathbf{c}(\mathbf{x})$  replacing  $u(\mathbf{x})$ . Then, there exist  $k_1$  and  $k_2$  such that

$$\tilde{E}[\|\theta_t - \hat{\theta}_t\|] \leq k_1 h^2 + \frac{k_2}{N^{1/2}}, \quad t \in [0, t_f] \quad (20)$$

where  $\hat{\theta}_t$  is the estimate of  $\theta_t$ , and  $N$  and  $h$  are the number of particles and the time step, respectively.

**Proof:** Let  $\theta_0$  be the initial condition for  $\theta$ . At  $t = 0$ ,  $N$  independent initial conditions are generated based on the density  $p(\mathbf{x}, \theta_0)$ , and the approximation method (18) is applied. From (19) we know that:

$$\tilde{E}\|E_{\theta_t} \mathbf{c}(\mathbf{x}_t) - \frac{1}{N} \sum_{i=1}^N \mathbf{c}(\hat{\mathbf{x}}_t^i)\| \leq Kh^2 + \frac{k'}{N^{1/2}}.$$

On the other hand, from (16), we know that  $\hat{\theta}$  is a solution to the system of equations

$$\frac{1}{N} \sum_{i=1}^N c_j(\hat{\mathbf{x}}_t^i) = E_{\hat{\theta}_t}(c_j(\mathbf{x}_t)), \quad \text{for } j = 1, \dots, p.$$

From Fact 5.2, the solution is exact if we replace  $\frac{1}{N} \sum_{i=1}^N c_j(\hat{\mathbf{x}}_t^i)$  by  $E_{\theta_t}(c_j(\mathbf{x}_t))$ . Subtracting the term  $E_{\theta_t}(c_j(\mathbf{x}))$  from both sides of the above equations and using the vector form for it, we get

$$\frac{1}{N} \sum_{i=1}^N \mathbf{c}(\hat{\mathbf{x}}_t^i) - E_{\theta_t}(\mathbf{c}(\mathbf{x}_t)) = E_{\hat{\theta}_t}(\mathbf{c}(\mathbf{x}_t)) - E_{\theta_t}(\mathbf{c}(\mathbf{x}_t)).$$

On the other hand, we know that  $E_\theta(\mathbf{c}(\mathbf{x}))$  is a differentiable and one to one function of  $\theta$  ( see Fact 5.2). The derivative of this function,  $g(\theta)$ , is positive definite and by assumption  $g(\theta) \geq \vartheta I$ . Therefore,  $\exists \alpha > 0$  such that

$$\begin{aligned} \|\theta_t - \hat{\theta}_t\| &\leq \alpha \|E_{\theta_t}(\mathbf{c}(\mathbf{x}_t)) - E_{\hat{\theta}_t}(\mathbf{c}(\mathbf{x}_t))\| \\ &= \alpha \|E_{\theta_t}(\mathbf{c}(\mathbf{x}_t)) - \frac{1}{N} \sum_{i=1}^N \mathbf{c}(\hat{\mathbf{x}}_t^i)\|. \end{aligned}$$

Taking the expectation on both sides of the inequality we have

$$\begin{aligned} \tilde{E}\|\theta_t - \hat{\theta}_t\| &\leq \alpha \tilde{E}\left\|\frac{1}{N} \sum_{i=1}^N \mathbf{c}(\hat{\mathbf{x}}_t^i) - E_{\theta_t}(\mathbf{c}(\mathbf{x}_t))\right\| \\ &\leq \alpha \left(Kh^2 + \frac{k'}{N^{1/2}}\right) \\ &= k_1 h^2 + \frac{k_2}{N^{1/2}} \end{aligned} \quad \diamond$$

Now we are ready to present the main result of this section.

**Theorem 5.6** For System (3) assume that  $\mathbf{f}_t(\cdot)$ ,  $G_t(\cdot)$ , and  $\mathbf{h}(\cdot)$  are such that for the Brownian motion  $\mathbf{w}_t$ , and the Gaussian noise  $\mathbf{v}_n$ , the conditional probability density of the state  $\mathbf{x}_t$ , conditioned on the observations, lies in the family  $\mathcal{S}$  for  $\Theta$  bounded and for  $t \in [0, T]$ . Also assume the conditions in Fact 5.2 and in Theorem 5.4 with  $\mathbf{c}(\mathbf{x})$  replacing  $u(\mathbf{x})$ . Then, if  $g^{-1}(\theta_t) E_{\theta_t}(\mathcal{L}_t \mathbf{c}(\mathbf{x}))$  is Lipschitz with Lipschitz constant  $L$  and  $g(\theta) \geq \vartheta I$ , there exist  $l_1$  and  $l_2$  such that

$$\tilde{E}\|\theta_n - \hat{\theta}_n\| \leq \sum_{i=0}^{n-1} \exp(Li\tau) \left( l_1 h^2 + \frac{l_2}{N^{1/2}} \right), \quad n\tau \in [0, T],$$

where  $\hat{\theta}_n$  is the estimate of  $\theta_n$ , and  $N$  and  $h$  are the number of particles and the time step, respectively. This inequality implies convergence of the estimated parameter,  $\hat{\theta}_n$ , to the true parameter,  $\theta_n$ , as  $h \rightarrow 0$  and  $N \rightarrow \infty$ .

**Proof:** Let  $\theta_t$  and  $\hat{\theta}_t$  be the actual and the estimated values of the parameter of the density at time  $t = n\tau$ , respectively. At time  $t' = (n+1)\tau$  the error in the estimate of  $\theta_{t'}$  is a combination of the error in the estimate in  $\hat{\theta}_t$  and the error added in the time interval  $[t, t']$ .

If the conditional density stays in the exponential family of densities,  $\theta_t$  has to satisfy the following differential equation:

$$\dot{\theta} = g^{-1}(\theta) E_{\theta_t}(\mathcal{L}_t \mathbf{c}(\mathbf{x})) dt, \quad n\tau \leq t < (n+1)\tau.$$

Let  $\tilde{\theta}_{t'}$  be the estimate of  $\theta_{t'}$ , if the error due to resampling and the approximation of the stochastic differential equation solution is not taken into account in the interval  $[t, t']$  (i.e.  $\tilde{\theta}_{t'}$  is computed from the above ordinary differential equation starting at  $\hat{\theta}_t$ ), then

$$\|\theta_{t'} - \hat{\theta}_{t'}\| \leq \|\theta_{t'} - \tilde{\theta}_{t'}\| + \|\tilde{\theta}_{t'} - \hat{\theta}_{t'}\|.$$

By the assumption of the theorem,  $g^{-1}(\theta) E_{\theta_t}(\mathcal{L}_t \mathbf{c}(\mathbf{x}))$  is Lipschitz with Lipschitz constant  $L$ , then by continuity of the solution of the differential equation with respect to the initial condition [25], we know that

$$\|\theta_{t'} - \tilde{\theta}_{t'}\| \leq \|\theta_t - \hat{\theta}_t\| e^{L(t'-t)},$$

therefore,

$$\tilde{E}\|\theta_{t'} - \tilde{\theta}_{t'}\| \leq \tilde{E}\|\theta_t - \hat{\theta}_t\| e^{L(t'-t)}.$$

Also from the Lemma 5.5,  $\exists k_1(t')$  and  $k_2(t')$  such that

$$\tilde{E}[\|\tilde{\theta}_{t'} - \hat{\theta}_{t'}\|] \leq k_1(t')h^2 + \frac{k_2(t')}{N^{1/2}},$$

therefore,

$$\tilde{E}\|\theta_{t'} - \hat{\theta}_{t'}\| \leq \tilde{E}\|\theta_t - \hat{\theta}_t\| e^{L(t'-t)} + k_1(t')h^2 + \frac{k_2(t')}{N^{1/2}}.$$

The observation noise  $\mathbf{v}_n$  and the function  $\mathbf{h}(\cdot)$  are such that Bayes' Rule does not introduce any further error in the estimate of  $\hat{\theta}_{t'}$ . More precisely,  $\Psi_n(\mathbf{x})$  is assumed to be a member of  $\mathcal{S}$ . This

implies that after applying Bayes' Rule to  $p(\mathbf{x}, \theta_{t'})$  and  $p(\mathbf{x}, \hat{\theta}_{t'})$  parameters  $\theta_{t'}$  and  $\hat{\theta}_{t'}$  are shifted with the same vector and therefore,  $\|\theta_{t+'} - \hat{\theta}_{t+'}\| = \|\theta_{t'} - \hat{\theta}_{t'}\|$ . Here  $t^{+}'$  represents the time right after Bayes' correction. Therefore, starting from the initial condition  $\theta_0$  we get

$$\tilde{E}\|\theta_n - \hat{\theta}_n\| \leq \sum_{i=0}^{n-1} \exp(Li\tau) \left( l_1 h^2 + \frac{l_2}{N^{1/2}} \right), \quad n\tau \in [0, T]$$

where

$$l_i = \max_n k_i(n\tau), \quad n\tau \in [0, T], \quad i = 1, 2.$$

◊

Here, we would like to make a few remarks:

- The result of Theorem 5.6 can be easily extended to convergence in the mean square sense.
- The assumption that the probability density stays in the family of densities,  $\mathcal{S}$ , does not seem very realistic. But with our approach, we should be able to get the result in [12]. In fact, in [12] the evolution of the density is forced to stay in the family at every single moment. In our method, we only force the density to be in the family at the end of each full iteration, i.e. observation epoch. This allows the estimated density to be closer to the actual density.
- In [12] the observation equation is considered to be time invariant. Here, the time-varying nature of  $\mathbf{h}_n(\mathbf{x})$  does not complicate the algorithm. It surely affects the assumption that the density stays in the family, but as we explained earlier, this assumption is not realistic to begin with, and it will be dropped.
- If  $u(\cdot)$  is in  $\mathcal{F}$ , then

$$\lim_{\substack{N \rightarrow \infty \\ h \rightarrow 0}} \tilde{E} \|E_\theta u(\mathbf{x}) - E_{\theta^*} u(\mathbf{x})\| = 0.$$

This is a criterion similar to the one used in [8].

## 6 Projection Particle Filtering for Exponential Families of Densities

In this section, we drop the assumption that the conditional density of the state given the observation (6) lies in the exponential family of densities,  $\mathcal{S}$ . Also, we do not require that  $\Psi_n(\mathbf{x})$  is a member of  $\mathcal{S}$ . Instead we make other assumptions. First we need the following definition:

**Definition 6.1** *We say that a function  $u(\cdot)$  belongs to the class  $\mathcal{F}_{k\kappa}$ , written as  $u \in \mathcal{F}_{k\kappa}$ , for fixed  $k > 0$  and  $\kappa > 0$ , such that for all  $\mathbf{x} \in \mathbb{R}^n$ , the following inequality holds:*

$$\|u(\mathbf{x})\| \leq k (1 + \|\mathbf{x}\|^\kappa).$$

The next two assumptions are to guarantee the existence of an exponential density close to the true conditional density.

**A 6.2** For the density in (6) there exists an exponential family of densities  $\mathcal{S}$  such that  $\forall t \in [0, T]$ ,  $\forall u \in \mathcal{F}_{k\kappa}$   $\exists \theta_t^* \in \Theta^*$  and  $\epsilon > 0$  such that

$$\tilde{E}\|E_{p_t}(u(\mathbf{x})) - E_{\theta_t^*}(u(\mathbf{x}))\| \leq \epsilon, \quad (21)$$

where  $\Theta^*$  is convex<sup>4</sup> and compact.

**A 6.3** For  $\theta_{n-}^*$  in (A6.2) and  $\Psi_n(\mathbf{x})$ ,  $\exists \Psi_n^*(\mathbf{x})$  such that

$$p(\mathbf{x}, \theta) = \frac{p(\mathbf{x}, \theta_{n-}^*)\Psi_n^*(\mathbf{x})}{\int p(\mathbf{x}, \theta_{n-}^*)\Psi_n^*(\mathbf{x})d\mathbf{x}}$$

is in the family  $\mathcal{S}$  for some  $\theta \in \Theta^*$  and we have:

- $\forall \theta \in \Theta^*$  and  $\forall u(\cdot) \in \mathcal{F}_{k\kappa}$ ,  $\exists \epsilon > 0$  such that

$$\tilde{E}\left\|\frac{E_\theta \Psi_n(\mathbf{x})u(\mathbf{x})}{E_\theta \Psi_n(\mathbf{x})} - \frac{E_\theta \Psi_n^*(\mathbf{x})u(\mathbf{x})}{E_\theta \Psi_n^*(\mathbf{x})}\right\| \leq \epsilon.$$

- $\forall u(\cdot) \in \mathcal{F}_{k\kappa}$ ,  $\exists \epsilon > 0$  such that

$$\tilde{E}\left\|\frac{E_{\theta_{n-}^*} \Psi_n^*(\mathbf{x})u(\mathbf{x})}{E_{\theta_{n-}^*} \Psi_n^*(\mathbf{x})} - \frac{E_{p_{n-}} \Psi_n(\mathbf{x})u(\mathbf{x})}{E_{p_{n-}} \Psi_n(\mathbf{x})}\right\| \leq \epsilon.$$

From Assumption (A6.3) it is clear that if  $\Psi_n^*(\cdot)$  satisfies the requirements of the assumption then  $c\Psi_n^*(\cdot)$  satisfies the same requirements, where  $c$  is a positive constant. Therefore, without loss of generality we assume that  $\Psi_n^*(\cdot) = \exp(\bar{\alpha}^T \mathbf{c}(\cdot))$  for some  $\bar{\alpha} \in \mathcal{R}^p$ . Using Assumption (A6.2), we can state the following fact.

**Fact 6.4**  $\forall \theta_1, \theta_2 \in \Theta^*$  and  $\forall u \in \mathcal{F}_{k\kappa}$ ,  $\exists K_1, K_2$  positive such that

$$\|E_{\theta_1} u(\mathbf{x}) - E_{\theta_2} u(\mathbf{x})\| \leq K_1 \|\theta_1 - \theta_2\| \quad (22)$$

$$\|\theta_1 - \theta_2\| \leq K_2 \|E_{\theta_1} \mathbf{c}(\mathbf{x}) - E_{\theta_2} \mathbf{c}(\mathbf{x})\|. \quad (23)$$

**Proof:** To prove (22), define  $f_u(\theta) = E_\theta u(\mathbf{x})$  for  $u(\cdot) \in \mathcal{F}_{k\kappa}$ . Then

$$\frac{d}{d\theta_i} f_u(\theta) = E_\theta c_i(\mathbf{x})u(\mathbf{x}) - E_\theta c_i(\mathbf{x})E_\theta u(\mathbf{x}).$$

---

<sup>4</sup>It is easy to see that the assumption of convexity is very natural. Assume  $\theta_1, \theta_2 \in \Theta^*$  then  $\int \exp(\theta_i^T \mathbf{c}(\mathbf{x}))d\mathbf{x} \leq \infty$  for  $i = 1, 2$ . Therefore, using the Holder inequality we have

$$\begin{aligned} \int \exp((\alpha \theta_1^T + (1-\alpha) \theta_2^T) \mathbf{c}(\mathbf{x}))d\mathbf{x} &= \int (\exp(\theta_1^T \mathbf{c}(\mathbf{x})))^\alpha (\exp(\theta_2^T \mathbf{c}(\mathbf{x})))^{(1-\alpha)} d\mathbf{x} \\ &\leq \left( \int \left( (\exp(\theta_1^T \mathbf{c}(\mathbf{x})))^\alpha \right)^{1/\alpha} d\mathbf{x} \right)^\alpha \left( \int \left( (\exp(\theta_2^T \mathbf{c}(\mathbf{x})))^{1-\alpha} \right)^{1/(1-\alpha)} d\mathbf{x} \right)^{1-\alpha} \\ &= \left( \int \exp(\theta_1^T \mathbf{c}(\mathbf{x}))d\mathbf{x} \right)^\alpha \left( \int \exp(\theta_2^T \mathbf{c}(\mathbf{x}))d\mathbf{x} \right)^{1-\alpha} \\ &\leq \infty \end{aligned}$$

where  $0 < \alpha < 1$ .

Since  $\|u(\mathbf{x})\| \leq k(1 + \|\mathbf{x}\|^\kappa)$  and  $\theta \in \Theta^*$ , where  $\Theta^*$  is compact, then there exists a constant  $A$  such that

$$\left\| \frac{df_u(\theta)}{d\theta} \right\| \leq A \quad \forall u(\cdot) \in \mathcal{F}_{k\kappa} \quad \text{and} \quad \forall \theta \in \Theta^*.$$

Since  $\Theta^*$  is convex and compact, it is clear that  $\exists K_1$  independent of  $u(\cdot)$  such that  $f_u(\mathbf{x})$  is Lipschitz over  $\Theta^*$  with the Lipschitz constant  $K_1$  [25].

Inequality (23) follows from the fact that  $\Theta^*$  is compact and the Fisher information matrix  $g(\theta) > \vartheta I$  over  $\Theta^*$ .

◊

Denote the interior of the set  $\Theta^*$  by  $\Theta_{int}^*$ . For  $\Theta_{int}^*$  we can state the following fact.

**Fact 6.5** *The set*

$$\mathcal{A} = \left\{ \alpha : \int \exp(\alpha^T \mathbf{c}(\mathbf{x})) \exp(\theta^T \mathbf{c}(\mathbf{x})) d\mathbf{x} < \infty, \forall \theta \in \Theta_{int}^* \text{ and } \alpha \in \mathbb{R}^p \right\}$$

is closed.

**Proof:** Assume  $\mathcal{A}$  is not closed. Therefore, there exists a converging sequence  $\{\alpha_i\} \subset \mathcal{A}$  with converging point  $\bar{\alpha} \notin \mathcal{A}$ , then  $\exists \bar{\theta} \in \Theta_{int}^*$  such that

$$\int \exp(\bar{\alpha}^T \mathbf{c}(\mathbf{x})) \exp(\bar{\theta}^T \mathbf{c}(\mathbf{x})) d\mathbf{x} > M, \quad \forall M \in \mathcal{R}.$$

Since  $\Theta_{int}^*$  is an open set,  $\exists \epsilon > 0$  such that  $\mathcal{N}_\epsilon(\bar{\theta}) \in \Theta_{int}^*$ . Also, since  $\{\alpha_i\}$  is a converging sequence,  $\exists k > 0$  such that  $\alpha_k \in \mathcal{N}_\epsilon(\bar{\alpha})$ . This implies that  $\theta_1 \in \Theta_{int}^*$  where  $\theta_1 = \bar{\theta} + \bar{\alpha} - \alpha_k$ . Therefore,

$$\int \exp(\alpha_k^T \mathbf{c}(\mathbf{x})) \exp(\theta_1^T \mathbf{c}(\mathbf{x})) d\mathbf{x} < \infty.$$

On the other hand, we know that

$$\exp(\alpha_k^T \mathbf{c}(\mathbf{x})) \exp(\theta_1^T \mathbf{c}(\mathbf{x})) = \exp(\bar{\alpha}^T \mathbf{c}(\mathbf{x})) \exp(\bar{\theta}^T \mathbf{c}(\mathbf{x})).$$

This is a contradiction, therefore,  $\mathcal{A}$  is closed.

◊

The following lemma is one of the building blocks of the results of this section.

**Lemma 6.6** *For  $\theta_{n-}^*$  and  $\Psi_n^*(\mathbf{x})$  defined in (A6.3), and  $\forall u(\cdot) \in \mathcal{F}_{k\kappa}$ ,  $\exists$  positive numbers  $k_1, k_2, k_3, k_4$  independent of  $\theta_{n-}^*$  and  $\Psi_n^*(\mathbf{x})$ , such that  $\forall \theta_1, \theta_2 \in \Theta^*$  the following are true.*

- (a)  $k_1 \leq \|E_\theta \Psi_n^*(\mathbf{x})\| \leq k_2 \quad \forall \theta \in \Theta^*$ .
- (b)  $\|E_\theta \Psi_n^*(\mathbf{x}) u(\mathbf{x})\| \leq k_3 \quad \forall \theta \in \Theta^*$ .
- (c)  $\|E_{\theta_1} \Psi_n^*(\mathbf{x}) u(\mathbf{x}) - E_{\theta_2} \Psi_n^*(\mathbf{x}) u(\mathbf{x})\| \leq k_4 \|\theta_1 - \theta_2\|$ .

**Proof:** Let  $\mathcal{M}$  be a set defined as follows

$$\mathcal{M} = \{\mathbf{m} : \mathbf{m} = \theta_1 - \theta_2, \quad \forall \theta_1, \theta_2 \in \Theta^*\}.$$

We claim that  $\mathcal{M}$  is compact. To prove this claim, assume  $\{\mathbf{m}_i\}$  to be a sequence in  $\mathcal{M}$ , i.e.  $\mathbf{m}_i \in \mathcal{M}$ . Also we assume that  $\lim_{i \rightarrow \infty} \mathbf{m}_i = \bar{\mathbf{m}}$ . We know that there exist sequences  $\{\theta_{1,i}\}$  and  $\{\theta_{2,i}\}$  such that  $\mathbf{m}_i = \theta_{1,i} - \theta_{2,i}$  and  $\theta_{1,i}, \theta_{2,i} \in \Theta^*$ . Since  $\Theta^*$  is compact there exist converging subsequences  $\{\bar{\theta}_{1,i}\}$  and  $\{\bar{\theta}_{2,i}\}$  in  $\Theta^*$ . This implies that  $\bar{\mathbf{m}} = \bar{\theta}_1 - \bar{\theta}_2$ , where  $\bar{\theta}_1$  and  $\bar{\theta}_2$  are the limits of the subsequences  $\{\bar{\theta}_{1,i}\}$  and  $\{\bar{\theta}_{2,i}\}$ . Since  $\bar{\theta}_1$  and  $\bar{\theta}_2 \in \Theta^*$ , then  $\bar{\mathbf{m}} \in \mathcal{M}$ , therefore  $\mathcal{M}$  is closed. Since  $\Theta^*$  is bounded,  $\mathcal{M}$  is bounded and therefore, it is compact.

We define set  $\mathcal{A}_1$  as follows:

$$\mathcal{A}_1 = \left\{ \alpha : \int \exp(\alpha^T \mathbf{c}(\mathbf{x})) \exp(\theta^T \mathbf{c}(\mathbf{x})) d\mathbf{x} < \infty, \forall \theta \in \Theta^* \text{ and } \alpha \in \mathbb{R}^p \right\}.$$

It is clear that  $\mathcal{A}_1 \subset \mathcal{A}$ . As we mentioned before, without loss of generality we can assume  $\Psi_n^*(\mathbf{x}) = \exp(\bar{\alpha}^T \mathbf{c}(\mathbf{x}))$  and from Assumption (A6.3) it is clear that  $\bar{\alpha} \in \mathcal{A} \cap \mathcal{M}$ . Since  $\mathcal{A} \cap \mathcal{M}$  and  $\Theta^*$  are compact we have

$$\min_{\theta \in \Theta^*} \min_{\alpha \in \mathcal{A} \cap \mathcal{M}} \|E_\theta \Psi_n^*(\mathbf{x})\| \leq \|E_\theta \Psi_n^*(\mathbf{x})\| \leq \max_{\theta \in \Theta^*} \max_{\alpha \in \mathcal{A} \cap \mathcal{M}} \|E_\theta \Psi_n^*(\mathbf{x})\|.$$

In other words (a) is true with  $k_1 = \min_{\theta \in \Theta^*} \min_{\alpha \in \mathcal{A} \cap \mathcal{M}} \|E_\theta \Psi_n^*(\mathbf{x})\|$  and

$k_2 = \max_{\theta \in \Theta^*} \max_{\alpha \in \mathcal{A} \cap \mathcal{M}} \|E_\theta \Psi_n^*(\mathbf{x})\|$ . Similarly, since  $u(\cdot) \in \mathcal{F}_{k\kappa}$ , (b) is true.

Using the above argument and the argument in Fact 6.4, we can show that  $\|\frac{d}{d\theta} E_\theta \Psi_n^*(\mathbf{x}) u(\mathbf{x})\|$  is uniformly bounded and since  $\Theta^*$  is convex and compact, then (c) is true [25].

◊

In the following we go through the proof of the theorem that we state later precisely. Assume  $\hat{\theta}_n$  is calculated according to Algorithm 5.1 and assume  $p(\mathbf{x}, \hat{\theta}_n)$  is such that  $\forall u \in \mathcal{F}_{k\kappa}$

$$\tilde{E} \|E_{\hat{\theta}_n} u(\mathbf{x}) - E_{\theta_n^*} u(\mathbf{x})\| \leq \delta, \quad (24)$$

where  $\theta_n^*$  (see Assumption (A6.2)) satisfies

$$\tilde{E} \|E_{p_n} u(\mathbf{x}) - E_{\theta_n^*} u(\mathbf{x})\| \leq \epsilon. \quad (25)$$

Using the density  $p(\mathbf{x}, \hat{\theta}_n)$ , new particles  $\mathbf{x}_n^1, \dots, \mathbf{x}_n^N$  are generated. The approximate solution for the stochastic differential equation at time  $(n+1)\tau$  maps these particles to  $\hat{\mathbf{x}}_{n+1}^1, \dots, \hat{\mathbf{x}}_{n+1}^N$ . From these new particles  $\hat{\theta}_{n+1}$  is calculated. From (24) and (25) we have

$$\tilde{E} \|E_{p_n} u(\mathbf{x}) - E_{\hat{\theta}_n} u(\mathbf{x})\| \leq \delta + \epsilon. \quad (26)$$

We define the function  $\mathbf{r}(\mathbf{x})$  as follows:

$$\mathbf{r}(\mathbf{x}) = E'' \mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)))$$

where  $\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau)$  is the approximate solution of stochastic differential equation (17) at time  $(n+1)\tau$  with the given initial condition  $\mathbf{x}$  at time  $n\tau$  using the method in (18). Since according to our assumption  $\mathbf{c} \in \mathcal{F}_{k\kappa}$ , then by using lemma 9.1 in [24], we have

$$\|\mathbf{r}(\mathbf{x})\| \leq K_3(1 + \|\mathbf{x}\|^\mu),$$

where  $K_3$  and  $\mu$  only depend on the function  $\mathbf{c}(\cdot)$  and the dimension of  $\mathbf{x}$ . We assume that  $\mathbf{r} \in \mathcal{F}_{kk}$ . If the argument of  $\mathbf{r}(\cdot)$  is a random variable, then using (26) we have

$$\tilde{E}\|E_{p_n}\mathbf{r}(\mathbf{x}) - E_{\hat{\theta}_n}\mathbf{r}(\mathbf{x})\| \leq \delta + \epsilon. \quad (27)$$

More explicitly,

$$\tilde{E}\|E_{p_n}E''[\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau))] - E_{\hat{\theta}_n}E''[\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau))]\| \leq \delta + \epsilon. \quad (28)$$

From Theorem 5.4 we have

$$\tilde{E}\|E_{p_n}\mathbf{c}(\mathbf{x}_{n,\mathbf{x}}((n+1)\tau)) - E_{p_n}E''\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau))\| \leq K_4 h^2, \quad (29)$$

for some  $K_4 > 0$ .

Using the Monte Carlo method to calculate the  $E_{p_n}\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau))$  brings another error term that is due to the finite number of particles as the initial conditions for method (18). The expectation of this error is bounded, i.e.  $\exists K_5 > 0$  s.t.

$$\tilde{E}\|E_{\hat{\theta}_n}E''\mathbf{c}(\hat{\mathbf{x}}_{n,\mathbf{x}}((n+1)\tau)) - \frac{1}{N} \sum_{i=1}^N \mathbf{c}(\hat{\mathbf{x}}_{n,\hat{\mathbf{x}}_i^i}((n+1)\tau))\| \leq \frac{K_5}{N^{\frac{1}{2}}}, \quad (30)$$

where  $\hat{\mathbf{x}}^i$  are distributed according to  $p(\mathbf{x}, \hat{\theta}_n)$ . Combining (28), (29), and (30) we get

$$\begin{aligned} \tilde{E}\|E_{p_n}\mathbf{c}(\mathbf{x}_{n,\mathbf{x}}((n+1)\tau)) - \frac{1}{N} \sum_{i=1}^N \mathbf{c}(\hat{\mathbf{x}}_{n,\hat{\mathbf{x}}_i^i}((n+1)\tau))\| &\leq \\ \delta + \epsilon + K_4 h^2 + \frac{K_5}{N^{\frac{1}{2}}}. \end{aligned} \quad (31)$$

Based on (A6.2), we know that  $\exists \theta_{(n+1)^-}^*$  such that

$$\tilde{E}\|E_{p_{(n+1)^-}}\mathbf{c}(\mathbf{x}) - E_{\theta_{(n+1)^-}^*}\mathbf{c}(\mathbf{x})\| \leq \epsilon. \quad (32)$$

We know that, if  $\mathbf{x}$  (initial condition at time  $n\tau$ ) is distributed according to  $p_n(\mathbf{x})$ , then  $E_{p_{(n+1)^-}}\mathbf{c}(\mathbf{x}) = E_{p_n}\mathbf{c}(\mathbf{x}_{n,\mathbf{x}}((n+1)\tau))$ , therefore, from (31) and (32) we get

$$\tilde{E}\|E_{\theta_{(n+1)^-}^*}\mathbf{c}(\mathbf{x}) - \frac{1}{N} \sum_{i=1}^N \mathbf{c}(\hat{\mathbf{x}}_{n,\hat{\mathbf{x}}_i^i}((n+1)\tau))\| \leq \delta + 2\epsilon + K_4 h^2 + \frac{K_5}{N^{\frac{1}{2}}}. \quad (33)$$

Then  $\hat{\theta}_{(n+1)^-}$  given by Algorithm 5.1 satisfies the following inequality

$$\tilde{E}\|E_{\theta_{(n+1)^-}^*}\mathbf{c}(\mathbf{x}) - E_{\hat{\theta}_{(n+1)^-}}\mathbf{c}(\mathbf{x})\| \leq \delta + 2\epsilon + K_4 h^2 + \frac{K_5}{N^{\frac{1}{2}}}. \quad (34)$$

From (A6.3) we know that  $\exists \theta \in \Theta^*$  such that

$$\begin{aligned} \tilde{E}\left\| \frac{\frac{E_{\theta_{(n+1)^-}^*}\Psi_{n+1}^*(\mathbf{x})u(\mathbf{x})}{E_{\theta_{(n+1)^-}^*}\Psi_{n+1}^*(\mathbf{x})} - \frac{E_{p_{(n+1)^-}}\Psi_{n+1}(\mathbf{x})u(\mathbf{x})}{E_{p_{(n+1)^-}}\Psi_{n+1}(\mathbf{x})}} \right\| &= \tilde{E}\|E_\theta u(\mathbf{x}) - E_{p_{(n+1)}}u(\mathbf{x})\| \\ &\leq \epsilon. \end{aligned}$$

Since  $\theta$  satisfies the inequality in (A6.2), we can choose  $\theta_{(n+1)}^*$  to be  $\theta$ , i.e.  $\theta_{(n+1)}^* = \theta$ .

On the other hand we have

$$\begin{aligned}
\left\| E_{\theta_{(n+1)}^*} u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}} u(\mathbf{x}) \right\| &= \left\| \frac{E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x})}{E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x})} - \frac{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}(\mathbf{x}) u(\mathbf{x})}{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}(\mathbf{x})} \right\| \\
&\leq \left\| \frac{E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x})}{E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x})} - \frac{E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x})}{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x})} \right\| + \\
&\quad \left\| \frac{E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x})}{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x})} - \frac{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x})}{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x})} \right\| + \\
&\quad \left\| \frac{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x})}{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x})} - \frac{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}(\mathbf{x}) u(\mathbf{x})}{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}(\mathbf{x})} \right\|,
\end{aligned}$$

therefore,

$$\begin{aligned}
\left\| E_{\theta_{(n+1)}^*} u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}} u(\mathbf{x}) \right\| &\leq \\
&\frac{\|E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x})\|}{\|E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x})\| \|E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x})\|} \left\| E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x}) \right\| + \\
&\frac{1}{\|E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x})\|} \left\| E_{\theta_{(n+1)}^*} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x}) \right\| + \\
&\left\| \frac{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x}) u(\mathbf{x})}{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}^*(\mathbf{x})} - \frac{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}(\mathbf{x}) u(\mathbf{x})}{E_{\hat{\theta}_{(n+1)}} \Psi_{n+1}(\mathbf{x})} \right\|.
\end{aligned}$$

Using Lemma 6.6 and (A6.3) we get

$$\tilde{E} \|E_{\theta_{(n+1)}^*} u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}} u(\mathbf{x})\| \leq \frac{k_3 k_4 + k_1 k_4}{k_1^2} \tilde{E} \|\theta_{(n+1)}^* - \hat{\theta}_{(n+1)}\| + \epsilon.$$

Therefore, from (34) and Fact 6.4 we get

$$\tilde{E} \|\theta_{(n+1)}^* - \hat{\theta}_{(n+1)}\| \leq K_2 \left( \delta + 2\epsilon + K_4 h^2 + \frac{K_5}{N^{\frac{1}{2}}} \right).$$

This implies that,  $\exists \iota_1, \iota_2, \iota_3, \iota_4 > 0$  such that

$$\tilde{E} \|E_{\theta_{(n+1)}^*} u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}} u(\mathbf{x})\| \leq \iota_1 \delta + \iota_2 \epsilon + \iota_3 h^2 + \iota_4 N^{-\frac{1}{2}}.$$

The next theorem summarizes our result in this section.

**Theorem 6.7** *For the system (3) assume (A2.1), (A2.2), (A6.2), and (A6.3). We also assume the conditions in Fact 5.2 and in Theorem 5.4 with  $\mathbf{c}(\mathbf{x})$  replacing  $u(\mathbf{x})$ , and we assume  $\mathbf{r} \in \mathcal{F}_{k\kappa}$ . Then in Algorithm 5.1 with approximation (18), if  $\forall u(\cdot) \in \mathcal{F}_{k\kappa}$*

$$\tilde{E} \|E_{\hat{\theta}_n} u(\mathbf{x}) - E_{\theta_n^*} u(\mathbf{x})\| \leq \delta$$

then

$$\tilde{E}\|E_{\theta_{(n+1)}^*}u(\mathbf{x}) - E_{\hat{\theta}_{(n+1)}}u(\mathbf{x})\| \leq \iota_1\delta + \iota_2\epsilon + \iota_3h^2 + \iota_4N^{-\frac{1}{2}},$$

for some  $\iota_1, \iota_2, \iota_3, \iota_4 > 0$ .

In Theorem 6.7 only one step of Algorithm 5.1 is considered; it is straightforward to then use Theorem 6.7 repeatedly for the time interval  $[0, T]$ , where  $T = M\tau$ . In that case,  $\|E_{\hat{\theta}_0}u(\mathbf{x}) - E_{\theta_0^*}u(\mathbf{x})\| \leq \delta_0$ , then  $\exists \alpha_1, \alpha_2, \alpha_3$ , and  $\alpha_4$  positive such that

$$\tilde{E}\|E_{\theta_n^*}u(\mathbf{x}((n)\tau)) - E_{\hat{\theta}_n}u(\mathbf{x}((n)\tau))\| \leq \alpha_1\delta_0 + \alpha_2\epsilon + \alpha_3h^2 + \alpha_4N^{-1/2},$$

for  $0 \leq n \leq M$ .

## 7 Particle Filtering for Nonlinear Systems with Constant Integer Uncertainty

Consider the following nonlinear dynamics and observation

$$\begin{aligned} d\mathbf{x}_t &= \mathbf{f}_t(\mathbf{x}_t)dt + G_t(\mathbf{x}_t)d\mathbf{w}_t \\ \mathbf{y}_{n\tau} &= \mathbf{h}_n(\mathbf{x}(n\tau)) + J_n\mathbf{z} + \mathbf{v}_n \end{aligned}$$

where the assumptions and the dimensions for  $\mathbf{x}_t$ ,  $\mathbf{y}_{n\tau}$ ,  $\mathbf{w}_t$ , and  $\mathbf{v}_n$  are the same as in the previous sections. We assume that  $\mathbf{z}$  is a random integer vector, i.e.  $\mathbf{z} \in \mathcal{Z}^m$  and  $J_n$  has the proper dimension. Vector  $\mathbf{z}$  is assumed to be constant in time. This problem can be set up in discrete time as well. In this case, the system dynamics and the observation can be written as follows:

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{f}_n(\mathbf{x}_n) + G_n(\mathbf{x}_n)\mathbf{w}_n \\ \mathbf{y}_n &= \mathbf{h}_n(\mathbf{x}_n) + J_n\mathbf{z} + \mathbf{v}_n \end{aligned}$$

In both setups we assume that the integer uncertainty affects only some components of the observation, and other components are unaffected by  $\mathbf{z}$ . The affected components have associated noise components in  $\mathbf{v}_n$  that have considerably lower energy. In other words, the uncertain components of  $\mathbf{y}_{n\tau}$  (or equivalently  $\mathbf{y}_n$ ) would be considerably more accurate than the other components, if the integer ambiguities were known. This suggests that an accurate estimation of  $\mathbf{z}$  can increase the accuracy of the estimate of the state of the system significantly. With this explanation, our treatment of  $\mathbf{z}$  is clear. From the state dynamics and the observation equation we first estimate  $\mathbf{z}$  and then, with fixed  $\mathbf{z}$ , we use regular nonlinear filtering methods to estimate the state of the system  $\mathbf{x}_t$ .

We augment the state  $\mathbf{x}_t$  with the integer ambiguity  $\mathbf{z}$ . Having done that, the state dynamics and the observation have the following form:

$$\begin{aligned} d\begin{bmatrix} \mathbf{x}_t \\ \mathbf{z}_t \end{bmatrix} &= \begin{bmatrix} \mathbf{f}_t(\mathbf{x}_t) \\ 0 \end{bmatrix} dt + \begin{bmatrix} G_t(\mathbf{x}_t) \\ 0 \end{bmatrix} d\mathbf{w}_t \\ \mathbf{y}_{n\tau} &= \mathbf{h}_n(\mathbf{x}(n\tau)) + J_n\mathbf{z}(n\tau) + \mathbf{v}_n. \end{aligned} \tag{35}$$

We assume that the initial distribution of  $(\mathbf{x}_0^T, \mathbf{z}_0^T)^T$  is known. Now the state dynamics and the observation have the same form as was studied in Section (4). Therefore, we can apply particle

filtering to find the conditional probability distribution of the augmented state. This setup is a special case of the setup in Section (4). In (35) there is no state transition for  $\mathbf{z}_t$ , therefore, using particle filtering in its original form may not be the best option. Recall that in particle filtering we start with  $N$  i.i.d. particles distributed according to the initial distribution. In the resampling part the low probability particles die and the high probability particles produce many particles identical to themselves. Since  $\mathbf{z}_t$  does not change, the part of the particles associated to  $\mathbf{z}_t$  tends to cover smaller and smaller portions of the state space. In fact, the state space of the integer vectors is defined by the particles at the initial time. This problem can be overcome by modifying the algorithm mentioned in Section (4). In the new algorithm, Step 5 is changed in such a way that the particles are the addition of the original particles found by Algorithm 4.1, with a random vector. The modification is very important for the integer values, since the integers do not have a dynamics that is driven by a random input. In [9], a similar modification has been used for the regular nonlinear filtering setup. It seems that the convergence results given in [9] can be applied to the current case as well.

Based on the modified algorithm, we simulated a nonlinear filtering problem similar to the problem involved in the GPS system.

In a two dimensional space, three transmitters (imagine three pseudo satellites) are mounted on three known points (2000, 100000), (0, 100000), and (-2000, 100000). The moving object can measure its distance from these transmitters. For each pseudo satellite, two types of measurement are possible: One with high measurement noise and the other with low measurement noise. For the low measurement noise, though, there is an integer ambiguity. The dynamics of the moving object for this example is considered to be in discrete time and linear time invariant. The dynamics and observation equation is given as follows:

$$\begin{pmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{pmatrix}_{n+1} = \begin{pmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{pmatrix}_n + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}_n,$$

$$\begin{aligned} y_n^a &= \|\mathbf{x} - \mathbf{s}_i\| + v_n^a & i = 1, 2, 3 \\ y_n^b &= \|\mathbf{x} - \mathbf{s}_i\| + n_i + v_n^b & i = 1, 2, 3, \end{aligned}$$

where  $\mathbf{x} = (x_1, x_2)^T$ ,  $\mathbf{s}_i$  is the position of pseudo satellite  $i$  in two dimensional space,  $\Delta t = 0.1$  unit of time,  $n_i$  is the integer ambiguity of the pseudo satellite  $i$ , and  $\mathbf{w} = (w_1, w_2, w_3, w_4)^T$  and  $\mathbf{v} = (v_1^a, v_1^b, v_2^a, v_2^b, v_3^a, v_3^b)^T$  are zero mean white Gaussian noise process with covariance matrices  $\Sigma_{\mathbf{w}} = \text{diag}(1, 0.5, 1, 0.5)$  and  $\Sigma_{\mathbf{v}} = \text{diag}(5, 0.2, 5, 0.2, 5, 0.2)$ , respectively. In the simulation, it is assumed that the initial condition for the position is distributed in a square of size  $200 \times 200$  units squared, symmetric with respect to the origin.

In brief, the simulation can be separated into two parts, initialization and the full non-linear filtering. In the initialization part, we start with the initial probability distribution for  $(x_1, x_2)$  and from a series of observations, we find an estimate for the probability distribution of  $(v_1, v_2)$ . In this part, we do not use the dynamics of the moving object. Using our estimate for the probability distribution of  $(x_1, v_1, x_2, v_2)$  we find the distribution for the integer ambiguity. After this, the initialization is over, and the full non-linear filter is used. There are some minor numerical considerations that we would like to point out. In the Bayes step of the algorithm, the numbers are usually very small, and without proper scaling the original algorithm would not work. In the resampling part, one can use the law of large numbers and regenerate the particles based on their weight without generating random numbers that are time consuming. The result of the simulations

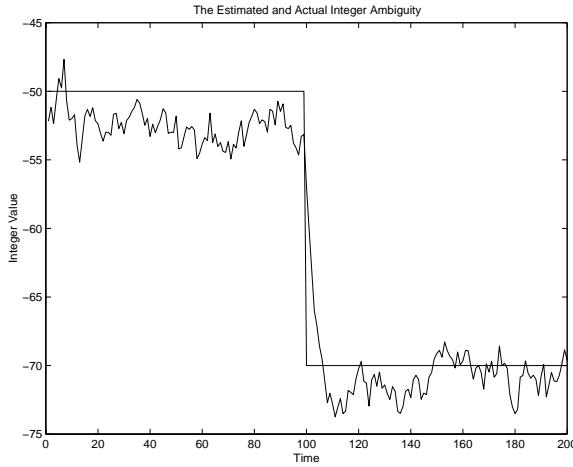


Figure 1: Estimated integer ambiguity versus the actual integer ambiguity of pseudo satellite (1). At time 100 there is a cycle slip of strength -20 for the measured phase of the carrier from pseudo satellite (1).

are shown in Figures 1, 2, 3, 4, 5, and 6. To display the estimated integers, we simply used the mean value, which is not necessarily the best choice. Of course, since we have the distribution, we can use the MAP estimate of the integers. In this simulation we forced one of the integers to have a jump. Although our algorithm is not designed for these kinds of changes, we see that it can estimate the new integer values. In future, we use special treatment for the times when these kinds of jumps happen. As we can see, the estimates for the integers are reasonably good. The reliability of the estimate for the integers depends on the energy of the noise.

## 8 Future Works

The simulations results show that our method is capable of estimating the integer ambiguity and the position. There are certain issues that need further investigation. In the following, we itemize these issues:

- What are the proper criteria to stop the integer ambiguity estimation part and fix the integers?
- What happens when a cycle slip happens, i.e. one or more of the integers have a jump? What change detection algorithm is proper and what is the performance of this algorithm? How can we repair the integer ambiguity efficiently?
- What happens when the number of the satellites drops from the critical number?
- How much improvement does the method of Section 5 for integer ambiguity and position estimation have over particle filtering?

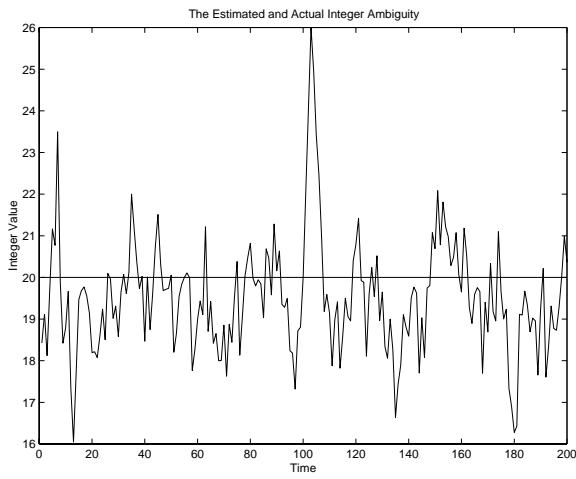


Figure 2: Estimated integer ambiguity versus the actual integer ambiguity of pseudo satellite (2). At time 100 there is a cycle slip of strength -20 for the measured phase of the carrier from pseudo satellite (1).

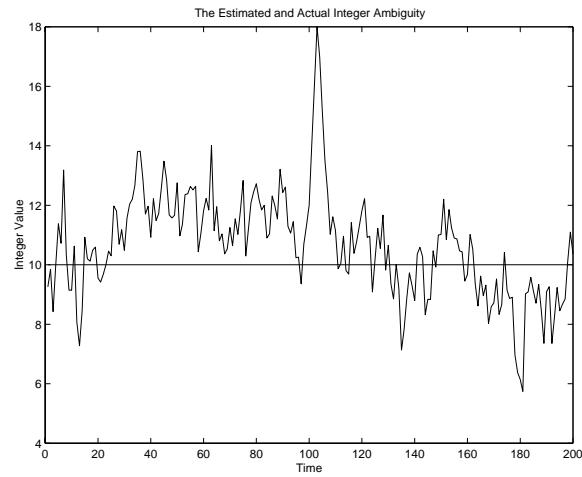


Figure 3: Estimated integer ambiguity versus the actual integer ambiguity of pseudo satellite (3). At time 100 there is a cycle slip of strength -20 for the measured phase of the carrier from pseudo satellite (1).

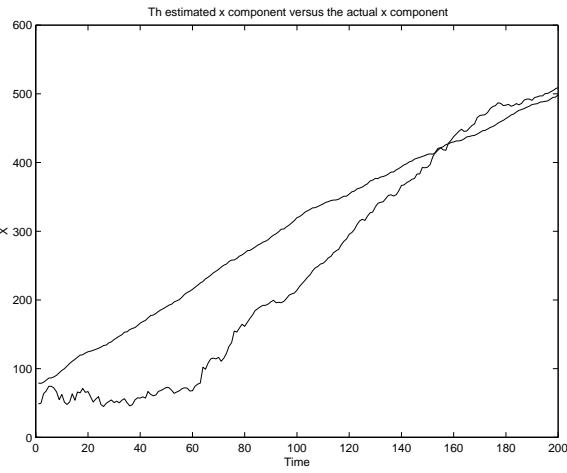


Figure 4: Estimated  $x_1$  component versus the actual  $x_1$  component of the position of the car. At time 100 there is a cycle slip of strength -20 for the measured phase of the carrier from pseudo satellite (1).

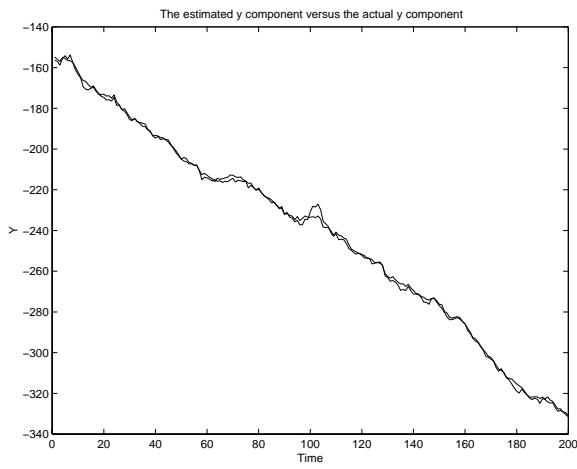


Figure 5: Estimated  $x_2$  component versus the actual  $x_2$  component of the position of the car. At time 100 there is a cycle slip of strength -20 for the measured phase of the carrier from pseudo satellite (1).

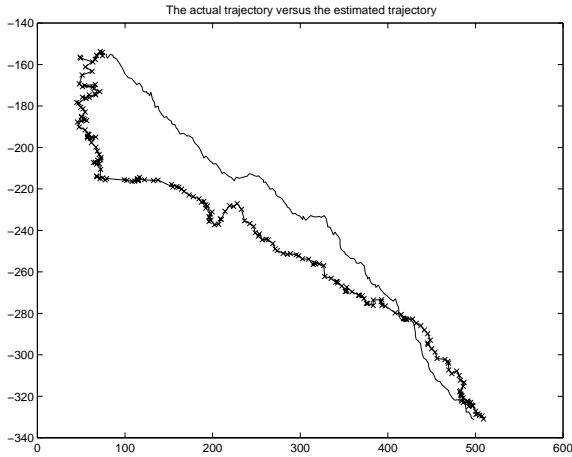


Figure 6: Estimated trajectory versus the actual trajectory of the car. At time 100 there is a cycle slip of strength -20 for the measured phase of the carrier from pseudo satellite (1).

These questions are to be answered in the future work. In addition to these, we shall be more specific in our simulations, and use real GPS data for our results.

## References

- [1] N. El-Sheimy and K.P. Schwarz. Navigating Urban Areas by VISAT-A Mobile Mapping System Integrating GPS/INS/Digital Camera for GIS Applications. *Navigation*, 45(4):275–285, Winter 1999.
- [2] R. Zickel and N. Nehemia. GPS Aided Dead Reckoning Navigation. *Proc. ION Nat. Tech. Meeting*, pages 577–586, 1994.
- [3] H. Carvalho, P. Del Moral, A. Monin, and G. Salut. Optimal Nonlinear Filtering in GPS/INS Integration. *IEEE Trans. on Aerospace and Electronic Systems*, 33(3):835–850, July 1997.
- [4] A. Hassibi and S. Boyd. Integer Parametric Estimation in Linear Models with Applications to GPS. *IEEE Trans. on Signal Processing*, 46(11):2938–52, Nov. 1998.
- [5] B. Hofmann-Wellenhof, H. Lichtenegger, and J. Collins. *Global Positioning System: Theory and Practice*. Springer-Verlag, Second edition, 1993.
- [6] P.J.G. Teunissen. A New Method for Fast Carrier Phase Ambiguity Estimation. *Proc. IEEE Position, Location and Navigation Symp.*, pages 562–73, 1994.
- [7] P.J.G. Teunissen. The Least Square Ambiguity Decorrelation Adjustment: a Method for Fast GPS Integer Ambiguity Estimation. *Journal of Geodesy*, 70(1-2):65–83, Nov. 1995.
- [8] P. Del Moral. Non Linear Filtering: Interacting Particle Solution. *Markov Processes and Related Fields*, 2(4):555–580, 1996.

- [9] F. LeGland, C. Musso, and N. Oudjane. An Analysis of Regularized Interacting Particle Methods in Nonlinear Filtering,. In J. Rojicek, M. Valeckova, M. Karry, and K. Warwick, editors, *Proceedings of the 3rd IEEE European Workshop on Computer-Intensive Methods in Control and Signal Processing*, volume 1, pages 167–174. Prague, September 7–9 1998.
- [10] P.M. Maybeck. *Stochastic Models, Estimation, and Control*, volume 2. Academic Press, 1982.
- [11] D. Brigo, B. Hanzon, and F. LeGland. A Differential Geometric Approach to Nonlinear Filtering : the Projection Filter. *Proceedings of the 34th IEEE Conference on Decision and Control*, pages 4006–4011, December 1995.
- [12] D. Brigo. *Filtering by Projection on the Manifold of Exponential Densities*. PhD thesis, Department of Economics and Econometrics, Vrije Universiteit, Amsterdam, October 1996.
- [13] D. Brigo and F. LeGland. A Finite Dimensional Filter with Exponential Conditional Density. *Proceedings of the 36th IEEE Conference on Decision and Control*, pages 1643–1644, December 10–12 1997.
- [14] R. Beard, J. Gunther, J. Lawton, and W. Stirling. Nonlinear Projection Filter Based on Galerkin Approximation. *Journal of Guidance Control, and Dynamics*, 22(2):258–266, March–April 1999.
- [15] M.H.A. Davis and S.I. Marcus. An Introduction to Nonlinear Filtering. In M. Hazewinkel and J.C. Willems edit. *Stochastic Systems: The Mathematics of Filtering and Identification and Applications*, pages 53–75, 1981.
- [16] K. Sobczyk. *Stochastic Differential Equation With Application to Physics and Engineering*. Kluwer Academic Publishers, 1991.
- [17] R.Z. Hasminskii. *Stochastic Stability of Differential Equations*. SIJTHOFF & NOORDHOFF Alphen aan den, The Netherlands, Rockville, Maryland, USA, 1980.
- [18] B.R. Crain. Exponential Models, Maximum Likelihood Estimation, and the Haar Condition. *Journal of the American Statistical Association*, 71(355):737–740, 1976.
- [19] S. Amari. *Differential-Geometrical Methods in Statistics*. Springer-Verlag, 1985.
- [20] N.J. Gordon, D.J. Salmond, and A.F.M. Smith. Novel Approach to Nonlinear/NonGaussian Bayesian State Estimation. *IEE Proceedings-F (Radar and Signal Processing)*, 140(2):107–113, 1993.
- [21] E.L. Lehmann and G. Casella. *Theory of Point Estimation*. Springer-Verlag, New York, second edition, 1998.
- [22] A.E. Gelfand and A.F.M. Smith. Sampling-Based Approaches to Calculating Marginal Densities. *Journal of the American Statistical Association*, 85(410):398–409, 1990.
- [23] P. Billingsley. *Probability and Measure*. John Wiley, New York, third edition, 1995.
- [24] G.N. Milstein. *Numerical Integration of Stochastic Differential Equations*. Kluwer Academic Publishers, New York, 1995.
- [25] H.K. Khalil. *Nonlinear Systems*. Printice-Hall, second edition, 1996.